Prerequisites for "Quantum channels I/II"

§ 0.1 Hilbert spaces and linear operators

complex vector space with inner product

Throughout the lecture, $\mathcal{H}$ denotes a finite-dimensional Hilbert space.

We use "bra-ket" notation: a <u>ket</u> is a vector $|\psi\rangle \in \mathcal{H}$

a <u>bra</u> is a vector $\langle\varphi| \in \mathcal{H}^*$

We have $\langle\psi| = (|\psi\rangle)^\dagger$, where $X^\dagger := \overline{X}^T$ denotes the Hermitian adjoint.

$B(\mathcal{H}_1, \mathcal{H}_2) := \{$linear maps from $\mathcal{H}_1$ to $\mathcal{H}_2\}$, $B(\mathcal{H}) = B(\mathcal{H}, \mathcal{H})$.

An operator $X \in B(\mathcal{H})$ is normal, if $XX^\dagger = X^\dagger X$. Every normal operator has a spectral decomposition: there exists a unitary U and a diagonal matrix D whose entries are the eigenvalues $\lambda_1, \dots, \lambda_d \in \mathbb{C}$ of X s.t. $X = UDU^\dagger$.

In other words, $X = \sum_{i=1}^{d} \lambda_i |\psi_i\rangle\langle\psi_i|$ where $\{|\psi_i\rangle\}_{i=1}^{d}$ are the eigenvectors of X, $X|\psi_i\rangle = \lambda_i |\psi_i\rangle$, and $U = (|\psi_1\rangle, \dots |\psi_d\rangle)$.

If X is Hermitian, $X^\dagger = X$, then $\lambda_i \in \mathbb{R}$.

$\langle\psi|X|\psi\rangle \geq 0 \quad \forall\, |\psi\rangle \in \mathcal{H}$

$\Updownarrow$

If X is positive semidefinite (PSD), $X \geq 0$, then $\lambda_i \geq 0$.

It holds that PSD $\Rightarrow$ Hermitian $\Rightarrow$ normal.

Unless stated otherwise, basis will always mean <u>orthonormal basis</u>.

§ 0.2 Quantum states

A quantum state ρ on a Hilbert space $\mathcal{H}$ is a PSD linear operator with unit trace: $\rho \in \mathcal{B}(\mathcal{H})$, $\rho \geq 0$, $\operatorname{tr}\rho = 1$.

This means that the eigenvalues $\{\lambda_i\}_{i=1}^{d}$ satisfy $\lambda_i \geq 0$ and $\sum_{i=1}^{d} \lambda_i = 1$, that is, $\{\lambda_i\}_i$ is a probability distribution.

A pure quantum state ψ is a quantum state of rank 1. We can find a vector $|\psi\rangle \in \mathcal{H}$ s.t. $\psi = |\psi\rangle\langle\psi|$, i.e. ψ is a projector.

A mixed state is a quantum state with rank > 1. Mixed states are convex combinations of pure states: for every quantum state ρ of rank $r = \operatorname{rk}\rho$ there are pure states $\{|\psi_i\rangle\}_{i=1}^{k}$ $(k \geq r)$ and a probability distribution $\{p_i\}_{i=1}^{k}$ s.t. $\rho = \sum_{i=1}^{k} p_i |\psi_i\rangle\langle\psi_i|$.

Special case: Spectral (or eigen-) decomposition

$$\rho = \sum_i \lambda_i |\varphi_i\rangle\langle\varphi_i| \quad \text{where } \rho|\varphi_i\rangle = \lambda_i|\varphi_i\rangle .$$

§ 0.3 Composite systems, partial trace, entanglement

Let A and B be two quantum systems with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$.

The joint system AB is described by the Hilbert space $\mathcal{H}_{AB} := \mathcal{H}_A \otimes \mathcal{H}_B$.

A quantum state on $\mathcal{H}_{AB}$ is often denoted ρ_{AB}.

The marginal ρ_A of a bipartite state ρ_{AB} is obtained via partial trace over B:

$$\rho_A := \mathrm{tr}_B\, \rho_{AB} \text{ defined via } \mathrm{tr}\left(\rho_{AB}\left(X_A \otimes \mathbb{1}_B\right)\right) = \mathrm{tr}\left(\rho_A X_A\right) \quad \forall X_A \in \mathcal{B}(\mathcal{H}_A)$$

Explicit formula for partial trace ($|B| := \dim \mathcal{H}_B$):

$$\mathrm{tr}_B\, \rho_{AB} = \sum_{i=1}^{|B|} \left(\mathbb{1}_A \otimes \langle i|_B\right) \rho_{AB} \left(\mathbb{1}_A \otimes |i\rangle_B\right)$$

for some basis $\{|i\rangle_B\}_{i=1}^{|B|}$ of $\mathcal{H}_B$.

A product state on AB is a state of the form $\rho_A \otimes \sigma_B$.

A state is called separable if it lies in the convex hull of product states:

$$\rho_{AB} = \sum_i p_i\, \rho_A^i \otimes \sigma_B^i$$

for some states $\{\rho_A^i\}_i$ and $\{\sigma_B^i\}_i$ and a prob. dist. $\{p_i\}_i$.

A state is called entangled, if it is not separable.

Maximally entangled state: Let $d = \dim \mathcal{H}$, $\{|i\rangle\}_{i=1}^d$ be a basis for $\mathcal{H}$,

$$|\phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle \in \mathcal{H} \otimes \mathcal{H}$$

§ 0.4 Measurements

The most general measurement is given by a positive operator-valued measure (POVM) $E = \{E_i\}_i$ where $E_i \geq 0$ $\forall i$ and $\sum_i E_i = \mathbb{1}$.

For a quantum system $\mathcal{H}$ in state ρ, the probability of obtaining measurement outcome i is given by $p_i = tr(\rho E_i)$.

Then, $\sum_i p_i = \sum_i tr(\rho E_i) = tr\left(\rho \sum_i E_i\right) = tr(\rho \mathbb{1}) = 1$ as required.

A projective measurement $\Pi = \{\Pi_i\}$ is a POVM where in addition the Π_i are orthogonal projectors: $\Pi_i \Pi_j = \delta_{ij} \Pi_i$.

Any basis $\{|e_i\rangle\}_{i=1}^{\dim \mathcal{H}}$ gives rise to a proj. mmn't $\Pi = \{|e_i\rangle\langle e_i|\}_{i=1}^{\dim \mathcal{H}}$

§ 0.5 Entropies

The Shannon entropy $H(p)$ of a probability distribution $p = \{p_1, \ldots, p_d\}$ is defined as $H(p) = -\sum_{i=1}^{d} p_i \log p_i$ $\quad (\log \equiv \log_2)$

The von Neumann entropy $S(\rho)$ of a quantum state ρ is defined as

$$S(\rho) = -tr(\rho \log \rho) = H(\{\lambda_1, \ldots, \lambda_d\})$$

where $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$ is a spectral decomposition of ρ.

Operator logarithm: $\log \rho = \sum_{i: \lambda_i > 0} \log(\lambda_i) |\psi_i\rangle\langle\psi_i|$.