$\boxed{\text{Recap}}$

·) Equality in data-processing: $D(\rho_{AB} \| \sigma_{AB}) = D(\rho_A \| \sigma_A)$

iff $\rho_{AB} = \sigma_{AB}^{1/2} \left(\sigma_A^{-1/2} \rho_A \sigma_A^{-1/2} \otimes \mathbb{1}_B\right) \sigma_{AB}^{1/2}$ $\quad (*)$

·) The channel $R_\sigma : A \to AB$, $R_\sigma(X_A) = \sigma_{AB}^{1/2} \left(\sigma_A^{-1/2} X_A \sigma_A^{-1/2} \otimes \mathbb{1}_B\right) \sigma_{AB}^{1/2}$

is known as the Petz recovery channel.

·) Application to quantum Markov chains ρ_{ABC}:

$I(A;C|B) = 0$ iff $D(\rho_{ABC} \| \rho_A \otimes \rho_{BC}) = D(\rho_{AB} \| \rho_A \otimes \rho_B)$

·) Prop 19, $(*)$: $\exists\, R : B \to BC$ such that $\rho_{ABC} = (id_A \otimes R)(\rho_{AB})$.

$\to$ ρ_{ABC} is a quantum Markov chain iff $I(A;C|B) = 0$.

·) Simple application: $D(\rho \| \sigma) = 0$ iff $\rho = \sigma$ for quantum states ρ, σ.

·) Generalization to arbitrary quantum channels:

$D(\rho \| \sigma) = D(\mathcal{N}(\rho) \| \mathcal{N}(\sigma))$ iff

$$\rho = \sigma^{1/2} \mathcal{N}^\dagger\left(\mathcal{N}(\sigma)^{-1/2} \mathcal{N}(\rho) \mathcal{N}(\sigma)^{-1/2}\right) \sigma^{1/2}$$

Petz recovery channel: $\quad R_{\sigma,\mathcal{N}}(X) := \sigma^{1/2} \mathcal{N}^\dagger\left(\mathcal{N}(\sigma)^{-1/2} X \mathcal{N}(\sigma)^{-1/2}\right) \sigma^{1/2}$

Application of equality condition in DPI to Holevo bound

·) RV $X \sim p_x \longrightarrow \{p_x, \rho_B^x\}$: $\rho_{XB} = \sum_x p_x |x\rangle\langle x|_X \otimes \rho_B^x$

·) POVM $M = \{M_y\}_y$, $\mathcal{M}(\omega_B) = \sum_y \operatorname{tr}(M_y \omega_B) |y\rangle\langle y|$

$M_y \geq 0$, $\sum_y M_y = \mathbb{1}_B$

·) $p_{y|x} := \operatorname{tr}(M_y \rho_B^x) \longrightarrow$ defines RV $(X,Y) \sim p_{xy} = p_{y|x} p_x$

·) Holevo bound : $\quad I_{acc}(\{p_x, \rho_B^x\}) = \max_{M \text{ POVM}} I(X;Y)$

$$\leq I(X;B)_\rho \quad \xleftarrow{} \text{DPI w.r.t. } id_X \otimes \mathcal{M}$$

$$= \chi(\{p_x, \rho_B^x\})$$

$$= S\Big(\sum_x p_x \rho_B^x\Big) - \sum_x p_x S(\rho_B^x)$$

$\boxed{\text{Prop 22}}$ We have equality in the Holevo bound iff the ensemble states commute pairwise:

$$\chi(\{p_x, \rho_B^x\}) = I(X;B) = I_{acc}(\{p_x, \rho_B^x\}) \text{ iff } [\rho_B^x, \rho_B^{x'}] = 0 \ \forall x, x'.$$

<u>Proof:</u> ($\Leftarrow$) $[\rho_B^x, \rho_B^{x'}] = 0 \ \forall x, x' \Rightarrow$ simultaneously diagonalize them

$\Rightarrow$ classical states $\Rightarrow I(X;B) = I(X;Y)$ ✓

⇐ POVM → projective measurements: Neumark's theorem

Let $M = \{M_y\}_y$ be a POVM on $\mathcal{H}_B$, then there exists $\mathcal{H}_E$,

an isometry $V: \mathcal{H}_B \to \mathcal{H}_E$, and a projective measurement

$N = \{N_y\}_y$ s.t. $\mathrm{tr}(M_y \omega_B) = \mathrm{tr}(N_y V \omega_B V^\dagger)$ $\forall y, \forall \omega_B$

POVM: $\cdot)$ $M_y \geq 0$ PVM: $\cdot)$ $N_y \geq 0$

$\cdot)$ $\sum_y M_y = \mathbb{1}_B$ $\cdot)$ $\sum_y N_y = \mathbb{1}_B$

$\cdot)$ $N_y N_{y'} = \delta_{yy'} N_y$

$\cdot)$ Prop 6 (iii): $D(\cdot \| \cdot)$ is invariant under isometries:

$$I(X;B) = D(\rho_{XB} \| \rho_X \otimes \rho_B) = D((\mathbb{1} \otimes V)\rho_{XB}(\mathbb{1} \otimes V)^\dagger \| \rho_X \otimes V \rho_B V^\dagger)$$

$\cdot)$ $[\rho_B^x, \rho_B^{x'}] = 0$ iff $[V\rho_B^x V^\dagger, V\rho_B^{x'} V^\dagger] = 0$.

⇒ w.l.o.g. $M = \{M_y\}_y$ is the projective measurement $(M_y M_{y'} = \delta_{yy'} M_y)$ achieving the max. in $I_{acc}(\{p_x, \rho_B^x\})$.

Holevo bound: $I(X;B) \geq I(X;Y)$ by DPI wrt $\mathrm{id}_X \otimes \mathcal{M}$,

$$\mathcal{M}(\omega) = \sum_y \mathrm{tr}(M_y \omega) |y\rangle\langle y|$$

Holevo bound: $I(X;B) \geq I(X;Y)$ by DPI wrt $id_X \otimes \mathcal{M}$

$$\mathcal{M}(\omega) = \sum_y tr(M_y \omega) |y\rangle\langle y|$$

$I(X;B) = D(\rho_{XB} \| \rho_X \otimes \rho_B)$ $\qquad \rho_{XB} = \sum_x p_x |x\rangle\langle x| \otimes \rho_B^x$

$I(X;Y) = D(\rho_{XY} \| \rho_X \otimes \rho_Y)$ $\qquad \rho_{XY} = (id_X \otimes \mathcal{M})(\rho_{XB})$

$$= \sum_x p_x |x\rangle\langle x|_X \otimes \mathcal{M}(\rho_B^x)$$

$$= \sum_x p_x |x\rangle\langle x|_X \otimes \sum_y \underbrace{tr(M_y \rho_B^x)}_{p_{y|x}} |y\rangle\langle y|_Y$$

$$= \sum_{x,y} \underbrace{p_x p_{y|x}}_{p_{xy}} |x\rangle\langle x|_X \otimes |y\rangle\langle y|_Y$$

$\rho_B = \sum_x p_x \rho_B^x = \bar{\rho}_B$

Prop 20: $D(\rho_{XB} \| \rho_X \otimes \rho_B) = D(\rho_{XY} \| \rho_X \otimes \rho_Y)$

$$\text{iff} \quad \rho_{XB} = \underbrace{(\rho_X \otimes \bar{\rho}_B)^{1/2}}_{\cancel{\rho_X^{1/2}} \otimes \bar{\rho}_B^{1/2}} (id \otimes \mathcal{M}^\dagger)\left(\underbrace{(\rho_X \otimes \rho_Y)^{-1/2}}_{\cancel{\rho_X^{-1/2}} \otimes \rho_Y^{-1/2}} \rho_{XY} \cancel{(\rho_X \otimes \rho_Y)^{-1/2}}\right) \quad (\cancel{\rho_X} \otimes \bar{\rho}_B)^{1/2}$$

(*)

Recall: $\mathcal{N} = \sum_i K_i \cdot K_i^\dagger \iff \mathcal{N}^\dagger = \sum_i K_i^\dagger \cdot K_i$

$$\mathcal{M}(\omega_B) = \sum_y \underbrace{\operatorname{tr}(M_y\,\omega_B)}\, |y\rangle\langle y| \qquad \operatorname{tr}(X) = \sum_i \langle i|X|i\rangle$$

for an ONB $\{|i\rangle\}_i$

$$= \operatorname{tr}(\underset{=M_y}{\underbrace{M_y^{1/2}}}\,\omega_B\, M_y^{1/2})$$

$$= \sum_y \operatorname{tr}(M_y\,\omega_B\, M_y)\, |y\rangle\langle y|$$

$$= \sum_{y,i} \langle i|M_y\,\omega_B\, M_y|i\rangle\, |y\rangle\langle y|$$

$$= \sum_{y,i} \underbrace{|y\rangle\langle i|M_y}_{K_{i,y}}\,\omega_B\,\underbrace{M_y|i\rangle\langle y|}_{K_{i,y}^\dagger} = \sum_{i,y} K_{i,y}\,\omega_B\, K_{i,y}^\dagger$$

$$\mathcal{M}^\dagger(\tau_y) = \sum_{i,y} K_{i,y}^\dagger\, \tau_y\, K_{i,y} = \sum_{i,y} M_y\, |i\rangle\langle y|\tau_y|y\rangle\langle i|\, M_y$$

$$= \sum_{i,y} \langle y|\tau_y|y\rangle\, M_y\, |i\rangle\langle i|\, M_y$$

$$\sum_i \cdot = \mathbb{1}$$

$$= \sum_y \langle y|\tau_y|y\rangle\, M_y$$

Insert this in (*):

$$\sum_x p_x\, |x\rangle\langle x| \otimes \rho_B^x = (\mathbb{1}\otimes \bar{\rho}_B^{1/2})\,(\mathrm{id}\otimes\mathcal{M}^\dagger)\,\big((\mathbb{1}\otimes\rho_y^{-1/2})\,\rho_{xy}\,(\mathbb{1}\otimes\rho_y^{-1/2})\big)\,(\mathbb{1}\otimes\bar{\rho}_B^{1/2})$$

$$\sum_x p_x |x\rangle\langle x| \otimes \rho_B^x = \left(\mathbb{1}\otimes \bar{\rho}_B^{1/2}\right)\left(id\otimes \mathcal{M}^\dagger\right)\Big(\underbrace{\left(\mathbb{1}\otimes \rho_y^{-1/2}\right)\rho_{xy}\left(\mathbb{1}\otimes\rho_y^{-1/2}\right)}\Big)\left(\mathbb{1}\otimes\bar{\rho}_B^{1/2}\right)$$

$$\left(\mathbb{1}_x\otimes \rho_y^{-1/2}\right)\rho_{xy}\left(\mathbb{1}\otimes\rho_y^{-1/2}\right) =$$

$$= \sum_x p_x |x\rangle\langle x| \otimes \sum_y p_y^{-1/2}\, p_y^{-1/2}\, p_{y|x} |y\rangle\langle y|$$

$\rho_y = \sum_y p_y |y\rangle\langle y|$

with $p_y = tr(M_y \bar{\rho}_B)$

$$= \sum_x p_x |x\rangle\langle x| \otimes \sum_y \frac{p_{y|x}}{p_y} |y\rangle\langle y|$$

$$\mathcal{M}^\dagger(\tau_y) = \sum_y \langle y|\tau_y|y\rangle M_y$$

$$\Rightarrow \left(id_x\otimes\mathcal{M}^\dagger\right)\left(\left(\mathbb{1}\otimes\rho_y^{-1/2}\right)\rho_{xy}\left(\mathbb{1}\otimes\rho_y^{-1/2}\right)\right)$$

$\in B(\mathcal{H}_B)$

$$= \sum_x p_x |x\rangle\langle x| \otimes \sum_y \frac{p_{y|x}}{p_y} M_y \quad \in B(\mathcal{H}_X\otimes\mathcal{H}_B)$$

Substitute in (*) to obtain:

$$\rho_{XB} = \sum_x p_x |x\rangle\langle x| \otimes \rho_B^x = \sum_x p_x |x\rangle\langle x| \otimes \sum_y \frac{p_{y|x}}{p_y} \bar{\rho}_B^{1/2} M_y \bar{\rho}_B^{1/2}$$

$$\Rightarrow \forall x, \quad \rho_B^x = \sum_y \frac{p_{y|x}}{p_y} \bar{\rho}_B^{1/2} M_y \bar{\rho}_B^{1/2}$$

$$\rho_B^x = \sum_y \frac{p_{y|x}}{p_y} \bar{\rho}_B^{1/2} M_y \bar{\rho}_B^{1/2} \quad . \qquad \text{To show: } [\rho_B^x, \rho_B^{x'}] = 0 \quad \forall x, x'$$

Note: $\bar{\rho}_B^{-1/2} \rho_B^x \bar{\rho}_B^{-1/2} = \sum_y \frac{p_{y|x}}{p_y} M_y$ is a spectral decomposition

orthogonal projectors

$$\boxed{\begin{aligned} p_{y|x} &= \mathrm{tr}(M_y \rho_B^x) \\ p_{xy} &= p_{y|x} p_x \\ p_y &= \sum_x p_{xy} = \mathrm{tr}\Big(M_y \underbrace{\sum_x p_x \rho_B^x}_{\bar{\rho}_B}\Big) \end{aligned}}$$

$$\Rightarrow \left(\bar{\rho}_B^{-1/2} \rho_B^x \bar{\rho}_B^{-1/2}\right)^{1/2} = \sum_y \frac{p_{y|x}^{1/2}}{p_y^{1/2}} M_y$$

$$\mathrm{tr}\left(\bar{\rho}_B \left(\bar{\rho}_B^{-1/2} \rho_B^x \bar{\rho}_B^{-1/2}\right)^{1/2}\right) = \sum_y \frac{p_{y|x}^{1/2}}{p_y^{1/2}} \underbrace{\mathrm{tr}(\bar{\rho}_B M_y)}_{p_y}$$

$$= \sum_y p_{y|x}^{1/2} p_y^{1/2}$$

$$= \sum_y \mathrm{tr}(M_y \rho_B^x)^{1/2} \mathrm{tr}(M_y \bar{\rho}_B)^{1/2}$$

$$= \sum_y \left(\langle M_y (\rho_B^x)^{1/2}, M_y (\rho_B^x)^{1/2} \rangle\right)^{1/2}$$

$$\times \left(\langle M_y \bar{\rho}_B^{1/2}, M_y \bar{\rho}_B^{1/2} \rangle\right)^{1/2}$$

Cauchy-Schwarz inequality

$$\geq \sum_y \left| \langle M_y (\rho_B^x)^{1/2}, M_y \bar{\rho}_B^{1/2} \rangle \right|$$

$$= \sum_y \left| \mathrm{tr}\left((\rho_B^x)^{1/2} M_y \bar{\rho}_B^{1/2}\right) \right|$$

$\sum_y M_y = \mathbb{1}$

$$\geq \mathrm{tr}\left((\rho_B^x)^{1/2} \bar{\rho}_B^{1/2}\right) \qquad (**)$$

$$\mathrm{tr}\left(\bar\rho_B\left(\bar\rho_B^{-1/2}\,\rho_B^x\,\bar\rho^{-1/2}\right)^{1/2}\right) \geq \mathrm{tr}\left((\rho_B^x)^{1/2}\,\bar\rho_B^{1/2}\right) \qquad (**)$$

RHS of $(**)$: $\mathrm{tr}\left((\rho_B^x)^{1/2}\left(\underbrace{\sum_x p_x\,\rho_B^x}\right)^{1/2}\right)$

$$\geq \sum_x p_x (\rho_B^x)^{1/2} \quad \text{by op. concavity of } t\mapsto t^{1/2}$$

$$\geq \sum_x p_x\, \underbrace{\mathrm{tr}\left((\rho_B^x)^{1/2}(\rho_B^x)^{1/2}\right)}_{\mathrm{tr}\,\rho_B^x = 1} \geq 1$$

LHS of $(**)$ $= \mathrm{tr}\left(\underbrace{\bar\rho_B^{1/2}\left(\bar\rho_B^{-1/2}\,\rho_B^x\,\bar\rho_B^{-1/2}\right)^{1/2}\bar\rho_B^{1/2}}_{\rho_B^x \# \bar\rho_B \ \dots\ \text{matrix geometric mean}}\right)$

$\left(= (\rho_B^x)^{1/2}(\bar\rho_B)^{1/2} \text{ if } [\rho_B^x, \bar\rho_B] = 0\right)$

satisfies $\rho_B^x \# \bar\rho_B \leq \frac{1}{2}(\rho_B^x + \bar\rho_B)$ (matrix version of geometric-arithmetic mean ineq.)

$\Rightarrow \mathrm{tr}\left(\rho_B^x \# \bar\rho_B\right) \leq \frac{1}{2}\,\mathrm{tr}\left(\rho_B^x + \bar\rho_B\right) = 1$

$\Rightarrow 1 \geq \mathrm{tr}\left(\bar\rho_B\left(\bar\rho_B^{-1/2}\,\rho_B^x\,\bar\rho_B^{-1/2}\right)^{1/2}\right) \geq \mathrm{tr}\left((\rho_B^x)^{1/2}\,\bar\rho_B^{1/2}\right) \geq 1$

$\Rightarrow$ equality in CS ineq.: $M_y(\rho_B^x)^{1/2} = \lambda_{xy} M_y \bar\rho_B^{1/2}$, $\lambda_{xy} \in \mathbb{C}$

in fact, $\lambda_{xy} \geq 0$ since $\mathrm{tr}\left(M_y(\rho_B^x)^{1/2}\right), \mathrm{tr}\left(M_y\bar\rho_B^{1/2}\right) \geq 0$.

$$M_y (\rho_B^x)^{1/2} = \lambda_{xy} M_y \bar{\rho}_B^{1/2}$$ $\longrightarrow$ sum over y and use $\sum_y M_y = \mathbb{1}_B$

$$\Rightarrow (\rho_B^x)^{1/2} \bar{\rho}_B^{-1/2} = \sum_y \lambda_{xy} M_y \quad \text{is Hermitian !}$$

$$\Rightarrow (\rho_B^x)^{1/2} \bar{\rho}_B^{-1/2} = \left((\rho_B^x)^{1/2} \bar{\rho}_B^{-1/2}\right)^\dagger = \bar{\rho}_B^{-1/2} (\rho_B^x)^{1/2} \quad \forall x$$

$$\Rightarrow \rho_B^x \bar{\rho}_B = \bar{\rho}_B \rho_B^x \quad \forall x$$

Recall: $\bar{\rho}^{-1/2} \rho_B^x \bar{\rho}^{-1/2} = \sum \frac{P_{y|x}}{p_y} M_y$ is a spectral decomposition

$$\bar{\rho}^{-1/2} \rho_B^x \bar{\rho}^{-1/2} \bar{\rho}^{-1/2} \rho_B^{x'} \bar{\rho}^{-1/2} = \bar{\rho}^{-1/2} \rho_B^{x'} \bar{\rho}^{-1} \rho_B^x \bar{\rho}^{-1/2}$$

$$\Rightarrow \rho_B^x \rho_B^{x'} = \rho_B^{x'} \rho_B^x \quad \forall x, x' \Rightarrow \text{claim.} \qquad \square$$