Recap

·) Entropic quantities:

-) Conditional entropy $S(A|B) = S(AB) - S(B)$

$$= -D(\rho_{AB} \| \mathbb{1}_A \otimes \rho_B)$$

-) Coherent information $I_c(A>B) = -S(A|B)$

-) Mutual information $I(A;B) = S(A) + S(B) - S(AB)$

$$= S(A) - S(A|B) = S(B) - S(B|A)$$

$$= D(\rho_{AB} \| \rho_A \otimes \rho_B)$$

·) Some properties:

-) Dimension bounds: $-\log|A| \leq S(A|B) \leq \log|A|$

$$0 \leq I(A;B) \leq 2 \log \min\{|A|, |B|\}$$

-) Data-processing: $S(A|BC) \leq S(A|C)$

($\Leftrightarrow$ SSA) $I(A;BC) \geq I(A;C) \mid I(AB;C) \geq I(A;C)$

-) Weak monotonicity: $S(A|B) + S(A|C) \geq 0$

-) Concavity: $S(A|B)_{\bar\rho} \geq \sum_i p_i \, S(A|B)_{\rho^i}$ for $\bar\rho = \sum_i p_i \, \rho^i_{AB}$

-) Holevo quantity: $\chi(\{p_x, \rho^x_A\}) = I(X;A)$ for $\rho_{XA} = \sum_x p_x |x\rangle\langle x|_X \otimes \rho^x_A$

-) Accessible information: Let $X \sim p_x$ be a RV, $\{\rho_B^x\}$ states, $E = \{E_B^y\}_y$ a POVM with $|Y| = |X|$ and Y the RV defined via $p(y|x) = \mathrm{tr}(E_B^y \rho_B^x)$ (i.e. $p(x,y) = p(y|x)\, p_x$).

Then $I_{acc}(\{p_x, \rho_B^x\}) := \max\limits_{E \text{ POVM}} I(X;Y)$

-) Holevo bound: $I_{acc}(\{p_x, \rho_B^x\}) \leq I(X;B)$

for $\rho_{XB} = \sum_x p_x |x\rangle\langle x|_X \otimes \rho_B^x$.

$\boxed{\text{Corollary 15}}$ n qubits can encode at most n classical bits.

Proof: Encoding classical information in qubits means to select quantum states $\{\rho_B^x\}$ such that measuring B reveals X.

$\rightarrow$ choosing a measurement (POVM) that defines a RV Y such that $I(X;Y)$ is maximal.

Holevo bound: $I_{acc}(\{p_x, \rho_B^x\}) \leq I(X;B) \overset{\text{Prop 14 i)}}{\leq} \log|B|$

n qubits: $B = (\mathbb{C}^2)^{\otimes n} \Rightarrow |B| = 2^n \Rightarrow \log|B| = n$.

The upper bound is achieved by encoding X in an ONB of $|B|$.

$\square$

d) Conditional mutual information

$[I(A;B) = S(A) + S(B) - S(AB)]$

$$I(A;B|C) = S(A|C) + S(B|C) - S(AB|C)$$
$$= S(AC) + S(BC) - S(C) - S(ABC) \quad (\geq 0 \text{ by SSA})$$
$$= S(A|C) - S(A|BC)$$
$$= I(AC;B) - I(C;B)$$

Operational interpretations: quantum state redistribution

erasure of correlations

quantum Markov chains (more later)

$\boxed{\text{Prop 16}}$ i) Dimension bound: $0 \leq I(A;B|C) \leq 2 \log \min\{|A|, |B|\}$

ii) Chain rule: $I(A;BC) = I(A;B) + I(A;C|B)$

iii) Classical conditioning: $I(A;B|X)_\rho = \sum_x p_x I(A;B)_{\rho^x}$

for qqc states $\rho_{ABX} = \sum_x p_x \rho_{AB}^x \otimes |x\rangle\langle x|_X$

iv) Duality: Let $|\psi\rangle_{ABCD}$ be pure, then

$$I(A;B|C)_\psi = I(A;B|D)$$

Proof: i) $I(A;B|C) = S(AC) + S(BC) - S(C) - S(ABC) \geq 0$ by SSA.

$$I(A;B|C) = I(AC;B) - \underbrace{I(C;B)}_{\geq 0} \leq I(AC;B) \underset{\text{Prop 14(i)}}{\leq} 2\log|B|$$

$$= I(A;BC) - I(A;C) \leq I(A;BC) \leq 2\log|A|$$

ii) $I(A;BC) = I(A;B) + I(A;C|B)$ → expand in $S(.)$ and check

iii) $\rho_{ABX} = \sum_x p_x \rho^x_{AB} \otimes |x\rangle\langle x|$:

$$I(A;B|X) = S(A|X) + S(B|X) - S(AB|X)$$

$$\overset{\text{Prop 12(vi)}}{=} \sum_x p_x \Big(\underbrace{S(A)_{\rho^x} + S(B)_{\rho^x} - S(AB)_{\rho^x}}_{I(A;B)_{\rho^x}} \Big)$$

iv) $I(A;B|C)_\psi = S(A|C) - S(A|BC)$ $\qquad$ [$|\psi\rangle_{RQT}$: $S(R|Q) = -S(R|T)$]

$|\psi\rangle_{ABCD}$

$$\overset{\text{Prop 12(ii)}}{=} -S(A|BD) + S(A|D)$$

$$= I(A;B|D)$$

$\square$

Central question for us: What does $I(A;B|C)=0$ mean?

Detour: classical Markov chains

Random variables X, Y, Z $(p(x,y,z))$ form a Markov chain,

$X \to Y \to Z$, if X and Z are independent when conditioned

on Y. $\iff$ $P_{XZ|Y} = P_{X|Y} \cdot P_{Z|Y}$ $\qquad \left(P_{X|Y} = \frac{P_{XY}}{P_Y} \right)$

$\iff$ $I(X;Z|Y)=0$ $\qquad \left(I(X;Z|Y) = \sum_y p_y \, I(X;Z)_{P_{XZ|y}} \right)$

$\iff$ $\exists$ stochastic matrix $W_{Z|Y}$ s.t.

$$\boxed{P_{XYZ} = W_{Z|Y} \, P_{XY}}$$

(Bayes thm: $P_{XY} = P_{X|Y} \cdot P_Y$

$P_{XYZ} = P_{XZ|Y} \cdot P_Y$

Markov condition: $P_{XZ|Y} = P_{X|Y} \cdot P_{Z|Y}$ $\quad | \cdot P_Y$

$$P_{XYZ} = P_{XZ|Y} \cdot P_Y = P_{Z|Y} \cdot \underbrace{P_{X|Y} \cdot P_Y}_{P_{XY}}$$)

$\boxed{\text{Def 17}}$ (Quantum Markov chain)

A tripartite quantum state ρ_{ABC} is called a quantum Markov chain $(A \to B \to C)$, if $\exists\, R: B \to BC$ s.t.

$$\rho_{ABC} = \left(id_A \otimes R\right)(\rho_{AB})$$

Informally: 'Can recover C from B'

$\boxed{\text{Thm 18}}$ ρ_{ABC} is a quantum Markov chain iff $I(A;C|B) = 0$.

Proof: ($\Rightarrow$) Let $R: B \to BC$ be the quantum channel satisfying

$$\rho_{ABC} = (id_A \otimes R)(\rho_{AB}). \quad \Rightarrow \rho_{BC} = R(\rho_B)$$

$$D(\rho_{ABC} \| \rho_A \otimes \rho_{BC}) \underset{tr_C}{\overset{DPI}{\geq}} D(\rho_{AB} \| \rho_A \otimes \rho_B)$$

$$\underset{id_A \otimes R}{\overset{DPI}{\geq}} D(\rho_{ABC} \| \rho_A \otimes \rho_{BC})$$

$$\Rightarrow D(\rho_{ABC} \| \rho_A \otimes \rho_{BC}) = D(\rho_{AB} \| \rho_A \otimes \rho_B)$$

$$\begin{array}{ccc} \| & & \| \\ -S(ABC) + S(A) + S(BC) & = & -S(AB) + S(A) + S(B) \end{array}$$

$$\Leftrightarrow \quad I(A;C|B) = 0$$