Recap

·) Entropic quantities are fundamental in information theory for characterizing info-processing tasks

·) Most of them can be derived from the relative entropy

$$D(\rho\|\sigma) = \operatorname{tr}\rho\,(\log\rho - \log\sigma)$$ (or generalizations thereof)

·) Von Neumann entropy: $S(\rho) = -\operatorname{tr}\rho\log\rho = -D(\rho\|\mathbb{1})$

·) A few operational interpretations: source compression

entanglement conversion

·) Some important mathematical properties:

-) Non-negativity: $0 \leq S(\rho) \leq \log\dim\mathcal{H}$ $\quad (\rho \in \mathcal{B}(\mathcal{H}))$

-) Concavity: $S\left(\sum_i \lambda_i \rho_i\right) \geq \sum_i \lambda_i S(\rho_i)$

-) Strong subadditivity: $S(AB) + S(BC) \geq S(ABC) + S(B)$

$\Leftrightarrow$ weak monotonicity: $S(A) + S(B) \leq S(AC) + S(BC)$

-) Monotonicity under unital quantum channels:

$$S(\rho) \leq S(\mathcal{N}(\rho))$$

b) Conditional entropy: $S(A|B)_\rho = S(AB)_\rho - S(B)_\rho$

$$= -D(\rho_{AB} \,\|\, \mathbb{1}_A \otimes \rho_B)$$

Operational interpretations: - source compression with side information

- state merging

$\boxed{\text{Prop 12}}$ i) Conditioning reduces entropy: $S(A|B) \leq S(A)$

ii) Duality relation: let ρ_{AB} have purification $|\rho\rangle_{ABC}$, then

$$S(A|B)_\rho = -S(A|C)_\rho$$

iii) $-\log|A| \leq S(A|B) \leq \log|A|$ $\quad \left(\pi_A = \frac{1}{|A|}\mathbb{1}_A\right)$

saturated for ϕ^+_{AB} $\quad$ satur. for $\pi_A \otimes \omega_B$

iv) Data-processing: $S(A|BC) \leq S(A|B)$ $\quad$ (SSA)

v) Weak monotonicity / monogamy of entanglement:

$$S(A|B) + S(A|C) \geq 0$$

("cannot be too entangled with two parties at the same time")

vi) Classical conditioning: let $\rho_{XA} = \sum_x p_x |x\rangle\langle x|_X \otimes \rho_A^x$, then

$$S(A|X) = \sum_x p_x \, S(A)_{\rho^x}$$

vii) Concavity: for $\bar{\rho} = \sum_i \lambda_i \rho^i_{AB}$: $S(A|B)_{\bar{\rho}} \geq \sum_i \lambda_i S(A|B)_{\rho^i}$

Proof: i), ii), iii) : last lecture

iv) Data-processing: $S(A|BC) \leq S(A|B)$

$$S(A|BC) \leq S(A|C)$$

$$S(ABC) - S(BC) \leq S(AB) - S(B) \quad \text{holds by SSA.}$$

v) Weak monotonicity: $S(A|B) + S(A|C) \geq 0$

$$S(AB) - S(B) + S(AC) - S(C) \geq 0 \quad \text{holds by WM.}$$

vi) $\rho_{XA} = \sum_x p_x |x\rangle\langle x|_X \otimes \rho_A^x \longrightarrow \rho_X = \sum_x p_x |x\rangle\langle x|$

$$S(A|X) = -D(\rho_{XA} \| \mathbb{1}_A \otimes \rho_X) = -D\Big(\sum_x p_x |x\rangle\langle x|_X \otimes \rho_A^x \,\Big\|\, \mathbb{1}_A \otimes \sum_x p_x |x\rangle\langle x|\Big)$$

$$\overset{\text{Prop 6 (iv)}}{=} -\sum_x p_x D(\rho_A^x \| \mathbb{1}_A)$$

$$= \sum_x p_x S(A)_{\rho^x}$$

vii) Concavity: $\bar{\rho}_{AB} = \sum_i \lambda_i \rho_{AB}^i \longrightarrow \bar{\rho}_B = \sum_i \lambda_i \rho_B^i$

$$S(A|B)_{\bar{\rho}} = -D\Big(\sum_i \lambda_i \rho_{AB}^i \,\Big\|\, \mathbb{1}_A \otimes \sum_i \lambda_i \rho_B^i\Big)$$

$$\underset{\text{Prop 6 (v)}}{\overset{\text{joint conv.}}{\geq}} \sum_i \lambda_i \big(-D(\rho_{AB}^i \| \mathbb{1}_A \otimes \rho_B^i)\big)$$

$$= \sum_i \lambda_i S(A|B)_{\rho^i} \qquad \square$$

$\boxed{\text{Cor 13}}$ ρ_{AB} separable $\Rightarrow$ $S(A|B)_\rho \geq 0$

Proof: ρ_{AB} SEP: $\rho_{AB} = \sum_i \lambda_i \, \omega_A^i \otimes \sigma_B^i$

$$S(A|B)_\rho \overset{\text{Prop 12(vii)}}{\geq} \sum_i \lambda_i \, S(A|B)_{\omega^i \otimes \sigma^i} \geq 0$$

$$S(A|B)_{\omega^i \otimes \sigma^i} = \underbrace{S(AB)}_{\substack{\text{product state:} \\ S(A)+S(B)}} - S(B) = S(A) + \cancel{S(B)} - \cancel{S(B)} = S(A) \geq 0$$

□

The converse does not hold, since there are bound entangled states ρ_{AB} for which $S(A|B)_\rho \geq 0$.

(Bound entangled state: entangled state that is undistillable, see lecture Q(1))

c) Coherent information: $I_c(A > B)_\rho = - S(A|B)_\rho = S(B) - S(AB)$

$$= D(\rho_{AB} \| \mathbb{1}_A \otimes \rho_B)$$

Operational interpretation: Entanglement distillation

Quantum information transmission

Quantum error correction

Hashing inequality: ρ_{AB} with $I_c(A>B)_\rho > 0$ is distillable

$\Leftrightarrow$ weak monotonicity: $S(A|B) + S(A|C) \geq 0$

$$\Leftrightarrow \quad I_c(A>B) + I(A>C) \leq 0 \quad (\sim \text{no-cloning})$$

d) Mutual information: $I(A;B)_\rho = S(A) + S(B) - S(AB)$

$= S(A) - S(A|B) = S(B) - S(B|A)$

$= D(\rho_{AB} \| \rho_A \otimes \rho_B)$

"relative entropy distance from being a product state"

Operational interpretations: measure for total correlations in bipartite state (classical + quantum)

entanglement-assisted classical communication

classical communication cost in state merging

$\boxed{\text{Prop 14}}$ i) $0 \leq I(A;B)_\rho \leq 2 \log \min\{|A|, |B|\}$

$0 \leq I(X;B)_\rho \leq \log \min\{|X|, |B|\}$

ii) $I(A;BC) \geq I(A;B)$, $I(AB;C) \geq I(A;C)$

iii) Holevo information: quantum state ensemble $\{p_x, \rho_A^x\} = \mathcal{E}$

$$\chi(\mathcal{E}) = S\left(\sum_x p_x \rho_A^x\right) - \sum_x p_x S(\rho_A^x)$$

$$= I(X;A)_\rho \quad \text{where } \rho_{XA} = \sum_x p_x |x\rangle\langle x|_X \otimes \rho_A^x .$$

iv) Holevo bound:

Let $X \sim p(x)$ be a (classical) discrete random variable, $\{\rho_B^x\}$ quantum states, and let $E = \{E_B^y\}_y$ be a POVM $\left(E^y \geq 0, \sum_y E^y = \mathbb{1}_B\right)$.

Denote by $p(y|x) = \operatorname{tr}(E_B^y \rho_B^x)$ the conditional prob. dist. defining an RV Y. $\quad \left(p(x,y) = p(y|x) \cdot p(x) \right)$

Accessible information: $\quad I_{acc}(\{p_x, \rho_B^x\}) = \max_{E \text{ POVM}} I(X;Y)$

Then, $\quad I_{acc}(\{p_x, \rho_B^x\}) \leq \chi(\{p_x, \rho_B^x\}) = I(X;B)_\rho$

where $\rho_{XB} = \sum_x p_x |x\rangle\langle x|_X \otimes \rho_B^x$

<u>Proof</u>: i) $0 \leq I(A;B)_\rho \leq 2 \log \min\{|A|, |B|\}$

$I(A;B)_\rho = D(\rho_{AB} \| \rho_A \otimes \rho_B) \geq 0$ by Prop 6(iii)

$$I(A;B) = S(A) - \underbrace{S(A|B)}_{\text{Prop 12(iii): } \geq -\log|A|} \leq S(A) + \log|A| \leq 2\log|A|$$

same for $S(B) - S(B|A)$.

$S(A|B) = -\log|A|$ for ϕ^+_{AB} maximally entangled $(|A| \leq |B|)$

$$I(A;B)_{\phi^+} = \underset{= \log|A|}{S(A) + S(B)} - \underbrace{S(AB)}_{=0} = 2\log|A|$$

$\rho_{XA} = \sum_x p_x |x\rangle\langle x|_X \otimes \rho_A^x$: $I(X;A) \leq \log \min\{|X|, |A|\}$

$$I(X;A) = S(A) - \underbrace{S(A|X)}_{\text{Prop 12 (vi): } \sum_x p_x S(A)_{\rho^x} \geq 0} \leq S(A) \leq \log|A|$$

$$I(X;A) = S(X) - \underbrace{S(X|A)}_{\text{Cor 13: } \geq 0 \text{ because } \rho_{XA} \text{ is separable}} \leq S(X) \leq \log|X|$$

ii) $I(AB;C) \geq I(A;C)$

$$\overset{\shortparallel}{D(\rho_{ABC} \| \rho_{AB} \otimes \rho_C)} \underset{\mathrm{tr}_B}{\overset{\mathrm{DPI}}{\geq}} D(\rho_{AC} \| \rho_A \otimes \rho_C) = I(A;C)$$

iii) $I(X;A) = S(A) - S(A|X) = S\left(\sum_x p_x \rho_A^x\right) - \sum_x p_x S(\rho_A^x)$

$$= \chi(\{p_x, \rho_A^x\})$$

$\rho_{XA} = \sum_x p_x |x\rangle\langle x| \otimes \rho_A^x$

$\Rightarrow \rho_A = \sum_x p_x \rho_A^x$

iv) $I_{acc}(\{p_x, \rho_B^x\}) = \max_{E\ \mathrm{POVM}} I(X;Y) \leq I(X;B)_\rho$

POVM $E = \{E_B^y\}_y \longrightarrow$ measurement channel $\mathcal{M}(\rho) = \sum_y \mathrm{tr}(E^y \rho)|y\rangle\langle y|$

(quantum $\to$ classical)

$$I(X;B) = D(\rho_{XB} \| \rho_X \otimes \rho_B)$$

$$= D\Big(\sum_x p_x |x\rangle\langle x| \otimes \rho_B^x \,\Big\|\, \rho_X \otimes \sum_x p_x \rho_B^x\Big) \qquad = \sum_y \underbrace{\overbrace{\sum_x p(y|x)p(x)}^{=p(y)}}$$

$$\overset{\text{DPI}}{\underset{\text{wrt. } \mathcal{M}}{\geq}} D\Big(\sum_x p_x |x\rangle\langle x| \otimes \sum_y p(y|x)|y\rangle\langle y| \,\Big\|\, \rho_X \otimes \sum_{x,y} p_x p(y|x)|y\rangle\langle y|\Big)$$

$$= I(X;Y) \qquad \big(p(y,x) = p(y|x) \cdot p(x)\big)$$

$$\Rightarrow \quad I_{acc}(\{p_x, \rho_B^x\}) \leq I(X;B) \qquad \square$$