Recap

- Data-processing inequality: For any quantum channel $\mathcal{N}$,

$$D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \qquad (*)$$

- Isometric invariance: $D(\rho\|\sigma) = D(V\rho V^\dagger \| V\sigma V^\dagger)$ for all isometries V

- Since any channel can be written as $\mathcal{N}(X) = \mathrm{tr}_E\, VXV^\dagger$ for an isometry V, claim $(*)$ follows from $D(\rho_{AB}\|\sigma_{AB}) \geq D(\rho_A\|\sigma_A)$

- Main proof tools:
  - relative modular operators $\Delta^{X,Y} := X \cdot Y^{-1}$
  - operator convexity of $\eta(t) = t\log t$ and operator Jensen's inequality:

    f op. convex, V isometry: $\quad f(V^\dagger A V) \leq V^\dagger f(A) V$

- Define $\Delta_{AB} = \rho_{AB}\cdot\sigma_{AB}^{-1}$, $\Delta_A = \rho_A \cdot \sigma_A^{-1}$, then:

$$D(\rho_*\|\sigma_*) = \langle \sigma_*^{1/2}, \eta(\Delta_*)(\sigma_*^{1/2})\rangle \quad \text{for } * = AB, A$$

- The map $V: A \to AB$, $V(X_A) = (X_A\,\sigma_A^{-1/2} \otimes \mathbb{1}_B)\,\sigma_{AB}^{1/2}$ satisfies:
  - $V^\dagger V = \mathrm{id}_A$ (isometry)
  - $V(\sigma_A^{1/2}) = \sigma_{AB}^{1/2}$
  - $V^\dagger \Delta_{AB} V = \Delta_A \quad \Rightarrow \quad \eta(\Delta_A) = \eta(V^\dagger \Delta_{AB} V) \leq V^\dagger \eta(\Delta_{AB}) V$

2. Entropies and optimality in data-processing

§2.1 Entropic quantities

Entropies are fundamental quantities in information theory

a) von Neumann entropy: $S(\rho) = -\mathrm{tr}\,\rho\log\rho = -D(\rho\|\mathbb{1})$

$S(A)_\rho = S(\rho_A)$ where $\rho_A \in \mathcal{B}(\mathcal{H}_A)$ is a quantum state

Operational interpretations: - Source / data compression

- entanglement conversion of pure states

$\boxed{\text{Prop 11}}$ i) $0 \leq S(A)_\rho \leq \log|A|$

$S(A)_\rho = 0$ iff ρ_A is pure, $= \log|A|$ iff $\rho_A = \frac{1}{|A|}\mathbb{1}_A$

ii) Concavity: $S\left(\sum_i \lambda_i \rho_i\right) \geq \sum_i \lambda_i S(\rho_i)$

iii) Strong subadditivity: $\forall \rho_{ABC} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$

$$S(ABC) + S(C) \leq S(AC) + S(BC) \qquad (A \leftrightarrow B \leftrightarrow C)$$

$\Longleftrightarrow$ weak monotonicity:

$$S(A) + S(B) \leq S(AC) + S(BC)$$

$|C| = 1$: $S(AB) \leq S(A) + S(B)$ (subadditivity)

iv) Let $\mathcal{N}$ be a unital quantum channel ($\mathcal{N}(\mathbb{1}) = \mathbb{1}$).

Then, $S(\rho) \leq S(\mathcal{N}(\rho))$ for all $\rho \in \mathcal{B}(\mathcal{H})$.

In particular, let $\{\Pi_i\}_{i=1}^{n}$ be a projective measurement,

$$\left(\Pi_i \geq 0 \, , \; \Pi_i \Pi_j = \delta_{ij} \Pi_i \, , \; \sum_i \Pi_i = \mathbb{1} \right)$$

and $\mathcal{N}(X) = \sum_i \Pi_i X \Pi_i \implies S(\rho) \leq S(\mathcal{N}(\rho))$

For example, $S(\rho) \leq S(\mathrm{diag}(\rho))$

Proof: i) $\rho = \sum_i \lambda_i |i\rangle\langle i|$ spectral decomposition

$$S(\rho) = -\sum_i \lambda_i \log \lambda_i \quad . \quad \lambda_i \in [0,1]$$

$$\implies S(\rho) \geq 0$$

$S(\rho) = 0 \iff \exists! \, i_0 : \lambda_{i_0} = 1$ and $\lambda_j = 0$ for $j \neq i_0$

$$\iff \rho = |i_0\rangle\langle i_0|$$

$S(\rho) \leq \log|A|$: $\underline{D(\rho_A \| \frac{1}{|A|} \mathbb{1}_A)} = \mathrm{tr}\, \rho_A \Big(\log \rho_A - \underbrace{\log \tfrac{1}{|A|} \mathbb{1}_A}_{-\log|A| \cdot \mathbb{1}}\Big)$

$$= -S(\rho_A) + \log|A| \geq 0$$

$D(\rho_A \| \frac{1}{|A|} \mathbb{1}_A) = 0$ iff $\rho_A = \frac{1}{|A|} \mathbb{1}_A$

will be proved later!

ii) $S(\sum_i \lambda_i \rho_i) \geq \sum_i \lambda_i S(\rho_i)$:

$$S(\sum_i \lambda_i \rho_i) = -D(\sum_i \lambda_i \rho_i \| \mathbb{1}) = -D(\sum_i \lambda_i \rho_i \| \sum_i \lambda_i \mathbb{1})$$

$$\underset{\text{Prop 6(v)}}{\overset{\text{joint conv.}}{\geq}} \sum_i \lambda_i \underbrace{(-D(\rho_i \| \mathbb{1}))}_{= S(\rho_i)} \quad \checkmark$$

iii) to show: $S(ABC) + S(C) \leq S(AC) + S(BC)$

$$D(\rho_{ABC} \| \rho_A \otimes \rho_{BC}) = \operatorname{tr} \rho_{ABC} (\log \rho_{ABC} - \log(\rho_A \otimes \rho_{BC})) \quad (*)$$

if $[X, Y] = 0$, then $\log(XY) = \log X + \log Y$

for $\rho_A \otimes \rho_{BC} = (\rho_A \otimes \mathbb{1}_B)(\mathbb{1}_A \otimes \rho_{BC})$:

$$\log(\rho_A \otimes \rho_{BC}) = \log(\rho_A \otimes \mathbb{1}_{BC}) + \log(\mathbb{1}_A \otimes \rho_{BC})$$

$$= (\log \rho_A) \otimes \mathbb{1}_{BC} + \mathbb{1}_A \otimes \log \rho_{BC}$$

substitute in $(*)$: ... $= \underbrace{\operatorname{tr} \rho_{ABC} \log \rho_{ABC}}_{= -S(ABC)} \underbrace{- \operatorname{tr} \rho_{ABC} (\log \rho_A \otimes \mathbb{1}_{BC})}_{S(A)}$

$$\underbrace{- \operatorname{tr} \rho_{ABC} (\mathbb{1}_A \otimes \log \rho_{BC})}_{S(BC)}$$

$$= -S(ABC) + S(A) + S(BC)$$

$$D(\rho_{ABC} \| \rho_A \otimes \rho_{BC}) = -S(ABC) + S(A) + S(BC)$$

$$\geq \quad (\text{by data-processing w.r.t. } tr_B)$$

$$D(\rho_{AC} \| \rho_A \otimes \rho_C) = -S(AC) + S(A) + S(C)$$

weak monotonicity: $S(A) + S(B) \leq S(AC) + S(BC) \quad (**)$

$\rho_{ABC} \rightarrow$ purification $|\rho\rangle_{ABCD}$

By Schmidt decomposition, ρ_B and ρ_{ACD} have the same spectrum

$$\Rightarrow S(B) = S(ACD)$$

same for $S(BC) = S(AD)$

use in $(**)$: $S(A) + S(ACD) \leq S(AC) + S(AD)$

which is SSA for the state ρ_{ACD}

$$S(ABC) + S(C) \leq S(AC) + S(BC) \overset{|C|=1}{\Rightarrow} S(AB) \leq S(A) + S(B)$$

(subadditivity)

iv) $\mathcal{N}$ unital $(\mathcal{N}(\mathbb{1}) = \mathbb{1})$:

$$S(\rho) = -D(\rho \| \mathbb{1}) \leq -D(\mathcal{N}(\rho) \| \underbrace{\mathcal{N}(\mathbb{1})}_{=\mathbb{1}}) = S(\mathcal{N}(\rho))$$

$\square$

b) Conditional entropy: $S(A|B)_\rho = S(AB)_\rho - S(B)_\rho$

$$= -D(\rho_{AB} \,\|\, \mathbb{1}_A \otimes \rho_B)$$

Operational interpretations: - source compression with side information

- state merging

$\boxed{\text{Prop 12}}$ i) Conditioning reduces entropy: $S(A|B) \leq S(A)$

ii) Duality relation: Let ρ_{AB} have purification $|\rho\rangle_{ABC}$, then

$$S(A|B)_\rho = -S(A|C)_\rho$$

iii) $-\log|A| \leq S(A|B) \leq \log|A|$ $\qquad \left(\pi_A = \frac{1}{|A|}\mathbb{1}_A\right)$

saturated for ϕ^+_{AB} $\qquad$ satur. for $\pi_A \otimes \omega_B$

iv) Data-processing: $S(A|BC) \leq S(A|B)$ $\qquad$ (SSA)

v) Weak monotonicity / monogamy of entanglement:

$$S(A|B) + S(A|C) \geq 0$$

("cannot be too entangled with two parties at the same time")

vi) Classical conditioning: Let $\rho_{XA} = \sum_x p_x |x\rangle\langle x|_X \otimes \rho_A^x$, then

$$S(A|X) = \sum_x p_x \, S(A)_{\rho^x}$$

vii) Concavity: for $\bar{\rho} = \sum_i \lambda_i \rho^i_{AB}$: $S(A|B)_{\bar{\rho}} \geq \sum_i \lambda_i S(A|B)_{\rho^i}$

Proof: i) $S(A|B) \leq S(A) \iff S(AB) - S(B) \leq S(A)$ (SA)

ii) $S(A|B) = -S(A|C)$ for a state $|\psi\rangle_{ABC}$ purifying ρ_{AB}

by Schmidt decomposition: $S(AB) = S(C)$

$S(B) = S(AC)$

$$S(A|B) = S(AB) - S(B) = S(C) - S(AC) = -S(A|C)$$

iii) $-\log|A| \leq S(A|B) \leq \log|A|$

$$S(A|B) \overset{(i)}{\leq} S(A) \overset{\text{Prop II (i)}}{\leq} \log|A|$$

$$S(A|B) = -S(A|C) \quad (C \text{ purifies } \rho_{AB})$$

$$\geq -S(A)$$

$$\geq -\log|A|$$

$S(A|B) = \log|A|$: $\rho_{AB} = \pi_A \otimes \omega_B$, $S(A|B) = S(AB) - S(B)$

$$= S(A)_\pi + \cancel{S(B)_\omega} - \cancel{S(B)_\omega}$$

$$= \log|A|$$

$d = |A| \leq |B|$

$\downarrow$

$S(A|B) = -\log|A|$: $\underline{\rho_{AB} = \phi^+_{AB}}$, $|\phi^+\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_i |i\rangle_A |i\rangle_B$

$\mathrm{tr}_A\, \phi^+_{AB} = \pi_{B'}$ where $|B'| = d$

$$S(A|B) = \underbrace{S(AB)}_{=0} - \underbrace{S(B)}_{=\log d} = -\log|A|.$$