QUANTUM CHANNELS II: DATA-PROCESSING, RECOVERY CHANNELS, QUANTUM MARKOV CHAINS

0. Introduction: Quantum channels

·) $\mathcal{H}_A, \mathcal{H}_B$: finite-dim. Hilbert spaces

$\mathcal{B}(\mathcal{H}_A), \mathcal{B}(\mathcal{H}_B)$: algebras of linear op's acting on $\mathcal{H}_A, \mathcal{H}_B, \ldots$

·) A quantum state $\rho_A \in \mathcal{B}(\mathcal{H}_A)$ is a positive semi-definite op of unit trace:

$$\rho_A \geq 0 \, , \quad \operatorname{tr} \rho_A = 1$$

·) If $\operatorname{rk} \rho_A = 1$, then ρ_A is called a pure state, there exists $|\rho_A\rangle \in \mathcal{H}_A$, $\langle \rho_A | \rho_A \rangle = 1$, s.t. $\rho_A = |\rho_A\rangle\langle\rho_A|$.

·) A quantum channel is a linear map $\mathcal{N} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ that is

·) completely positive: $\mathcal{N} \otimes \mathrm{id}_n$ is positive $\forall n \in \mathbb{N}$

(CP) $\qquad$ (T positive: $(X \geq 0 \Rightarrow T(X) \geq 0)$,

$\mathrm{id}_n : \mathcal{B}(\mathbb{C}^n) \to \mathcal{B}(\mathbb{C}^n), \, X \mapsto X$)

·) trace-preserving: $\operatorname{tr} \mathcal{N}(X_A) = \operatorname{tr} X_A \quad \forall X_A \in \mathcal{B}(\mathcal{H}_A)$

(TP)

$\cdot)$ $\mathcal{N}: \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ is CPTP iff

$(\mathcal{N}: A \to B)$

$|B| \times |A|$ matrices

i) (Kraus representation): $\exists\, K_i : \mathcal{H}_A \to \mathcal{H}_B$ s.t.

$$\mathcal{N}(X_A) = \sum_i K_i X_A K_i^\dagger \qquad (CP)$$

and $\sum_i K_i^\dagger K_i = \mathbb{1}_A$.

ii) (isometric picture): $\exists$ isometry $V: \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$

for some auxiliary Hilbert space $\mathcal{H}_E$ ("environment") s.t.

$$\mathcal{N}(X_A) = \mathrm{tr}_E\, V X_A V^\dagger$$

(V is an isometry $\Leftrightarrow V^\dagger V = \mathbb{1}_A$,

$$\mathrm{tr}_E\, Y_{BE} = \sum_{i=1}^{|E|} (\mathbb{1}_B \otimes \langle i|_E)\, Y_{BE}\, (\mathbb{1}_B \otimes |i\rangle_E)$$

where $\{|i\rangle\}_{i=1}^{|E|}$ is an ONB for $\mathcal{H}_E$;

$|E| = \dim \mathcal{H}_E$.)

$\mathcal{H}_{E'} \cong \mathcal{H}_B$

iii) (unitary picture): $\exists$ unitary $U: \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_{E'} \to \mathcal{H}_{ABE'}$

and a state $|\varphi\rangle \in \mathcal{H}_B \otimes \mathcal{H}_{E'}$ s.t.

$$\mathcal{N}(X_A) = \mathrm{tr}_{AE'}\, U (X_A \otimes |\varphi\rangle\langle\varphi|_{BE'})\, U^\dagger$$

·) The Choi operator $\tau_{AB}^{\mathcal{N}}$ of a linear map $\mathcal{N}: A \to B$ is defined as

$$\tau_{AB}^{\mathcal{N}} = (id \otimes \mathcal{N})(\gamma_{AA'}),$$

where $|\gamma\rangle_{AA'} = \sum_{i=1}^{|A|} |i\rangle_A \otimes |i\rangle_{A'}$ $\left(\{|i\rangle\}_{i=1}^{|A|} \text{ is a basis for } \mathcal{H}_A \cong \mathcal{H}_{A'}\right)$

is an unnormalized maximally entangled state.

$\mathcal{N}: A \to B$ is ·) CP iff $\tau_{AB}^{\mathcal{N}} \geq 0$;

·) TP iff $\mathrm{tr}_B\, \tau_{AB}^{\mathcal{N}} = \mathbb{1}_A$.

$\tau_{AB}^{\mathcal{N}}$ uniquely defines $\mathcal{N}: A \to B$ via the Choi-Jamiołkowski

isomorphism: $\mathcal{N}(X_A) = \mathrm{tr}_A\left[\tau_{AB}^{\mathcal{N}}\left(X_A^T \otimes \mathbb{1}_B\right)\right] \quad \forall X_A \in \mathcal{B}(\mathcal{H}_A)$

·) Let $\mathcal{N}: A \to B$ be a linear map. The adjoint map $\mathcal{N}^\dagger: B \to A$

is defined via $\langle \mathcal{N}^\dagger(Y_B), X_A \rangle = \langle Y_B, \mathcal{N}(X_A) \rangle$

$\langle X, Y \rangle := \mathrm{tr}(X^\dagger Y)$ $\quad \forall X_A \in \mathcal{B}(\mathcal{H}_A), Y_B \in \mathcal{B}(\mathcal{H}_B)$

Hilbert-Schmidt inner product

·) A linear map $\mathcal{N}: A \to B$ is ·) CP iff $\mathcal{N}^\dagger$ is CP

·) TP iff $\mathcal{N}^\dagger$ is unital:

$$\mathcal{N}^\dagger(\mathbb{1}_B) = \mathbb{1}_A.$$

·) For a CP map $\mathcal{N} = \sum_i K_i \cdot K_i^\dagger$, the adjoint map is

$$\mathcal{N} = \sum_i K_i^\dagger \cdot K_i$$

Example: $\mathcal{N} = \mathrm{tr}_B : AB \to A$ $\qquad \{|i\rangle\}_{i=1}^{|B|}$ ONB for $\mathcal{H}_B$

$$\mathrm{tr}_B : X_{AB} \mapsto \sum_i \underbrace{(\mathbb{1}_A \otimes \langle i|_B)}_{=K_i} X_{AB} (\mathbb{1}_A \otimes |i\rangle_B)$$

$$\underline{\mathcal{N}^\dagger(Y_A)} = \sum_i K_i^\dagger Y_A K_i = \sum_i (\mathbb{1}_A \otimes |i\rangle_B) Y_A (\mathbb{1}_A \otimes \langle i|_B)$$

$$= \underline{Y_A \otimes \mathbb{1}_B} .$$

1. Relative entropy and data-processing

§1.1 Motivation: quantum state discrimination

Assume you're given one of two quantum states ρ_0, ρ_1 with equal probability $\frac{1}{2}$.

Goal: decide which one you get

Strategy: perform a measurement on the unknown state giving you a guess for the correct answer.

For us: measurement is described by a positive operator-valued measure (POVM): $\{\Lambda_i\}_{i=1}^{n}$, $\Lambda_i \geq 0$, $\sum_i \Lambda_i = \mathbb{1}_{\mathcal{H}}$

for a given state ρ, POVM gives the outcome "i" with probability $\mathrm{tr}(\Lambda_i \rho) = p_i$.

Here, we use a POVM $\Lambda = \{\Lambda_0, \Lambda_1\}$ $(\Lambda_1 = \mathbb{1} - \Lambda_0,\ \Lambda_0 \geq 0)$

outcome "0" corresponds to ρ_0 w.p. $\mathrm{tr}(\Lambda_0 \sigma)$

"1" —"— ρ_1 w.p. $\mathrm{tr}(\Lambda_1 \sigma)$

$\sigma \in \{\rho_0, \rho_1\}$

Success probability of correctly identifying the state:

$$P_{succ}(\Lambda) = \frac{1}{2} \underbrace{\Pr(\rho_0 | \rho_0)}_{\mathrm{tr}\,\Lambda_0 \rho_0} + \frac{1}{2} \underbrace{\Pr(\rho_1 | \rho_1)}_{\mathrm{tr}\,\Lambda_1 \rho_1}$$

$\max\limits_{\Lambda = \{\Lambda_0, \Lambda_1\}} \Leftrightarrow \max\limits_{0 \leq \Lambda_0 \leq \mathbb{1}}$

$$= \frac{1}{2} \left(\mathrm{tr}\,\Lambda_0 \rho_0 + \mathrm{tr}\,\Lambda_1 \rho_1 \right)$$

$\mathrm{tr}\,\Lambda_1 \rho_1 = \mathrm{tr}(\mathbb{1} - \Lambda_0)\rho_1 = 1 - \mathrm{tr}\,\Lambda_0 \rho_1$

$$= \frac{1}{2} \left(\mathrm{tr}\,\Lambda_0 \rho_0 + 1 - \mathrm{tr}\,\Lambda_0 \rho_1 \right)$$

$$= \frac{1}{2} \left(1 + \mathrm{tr}\,\Lambda_0 (\rho_0 - \rho_1) \right)$$

Want to max. $P_{succ}(\Lambda)$ with respect to POVM $\Lambda = \{\Lambda_0, \Lambda_1\}$

$$P^*_{succ} = \max_{0 \leq \Lambda_0 \leq \mathbb{1}} P_{succ}(\Lambda) = \frac{1}{2} \left(1 + \max_{0 \leq \Lambda_0 \leq \mathbb{1}} \mathrm{tr}\,\Lambda_0 (\rho_0 - \rho_1) \right)$$

$$P_{succ} = \max_{0 \le \Lambda \le \mathbb{1}} P_{succ}(\Lambda) = \frac{1}{2}\Big(1 + \underline{\max_{0\le\Lambda\le\mathbb{1}} \mathrm{tr}\,\Lambda(\rho_0 - \rho_1)}\Big)$$

- <u>trace norm</u> $\|X\|_1 = \mathrm{tr}\sqrt{X^\dagger X} = \sum_i s_i(X)$

  s_i: singular values of X

  $\Longleftrightarrow$ eigenvalues of $|X| = \sqrt{X^\dagger X}$

- <u>trace distance</u> of two states ρ_0, ρ_1: $\frac{1}{2}\|\rho_0 - \rho_1\|_1$

Lemma 1 Let ρ_0, ρ_1 be quantum states, then

$$\frac{1}{2}\|\rho_0 - \rho_1\|_1 = \max_{0\le\Lambda\le\mathbb{1}} \mathrm{tr}\,\Lambda(\rho_0 - \rho_1)$$

<u>Proof</u>: $\rho_0 - \rho_1$ is Hermitian: $\rho_0 - \rho_1 = \sum_i \lambda_i |i\rangle\langle i|$

where: $\lambda_i \in \mathbb{R}$ (eigenvalues), $\langle i|j\rangle = \delta_{ij}$ (eigenvectors).

Define:
$$\left.\begin{aligned} P &= \sum_{i:\lambda_i \ge 0} \lambda_i |i\rangle\langle i| \ge 0 \\ Q &= \sum_{i:\lambda_i < 0} (-\lambda_i)|i\rangle\langle i| \ge 0 \end{aligned}\right\} \quad \rho_0 - \rho_1 = P - Q$$

$$\|\rho_0 - \rho_1\|_1 = \mathrm{tr}\,|\rho_0 - \rho_1| = \mathrm{tr}\,|P - Q| = \mathrm{tr}\,P + \mathrm{tr}\,Q = 2\,\mathrm{tr}\,P$$

$|A| = \sqrt{A^\dagger A}$

$$\mathrm{tr}(P - Q) = \mathrm{tr}(\rho_0 - \rho_1) = \mathrm{tr}\,\rho_0 - \mathrm{tr}\,\rho_1 = 0$$

$$\Rightarrow \mathrm{tr}\,P = \mathrm{tr}\,Q$$

$$\Rightarrow \frac{1}{2} \underline{\|\rho_0 - \rho_1\|_1} = \mathrm{tr}\, P$$

relate this to $\mathrm{tr}\, \Lambda(\rho_0 - \rho_1)$ for arbitrary $0 \le \Lambda \le \mathbb{1}$:

$$\mathrm{tr}\, \Lambda(\rho_0 - \rho_1) = \mathrm{tr}\, \Lambda(P - Q)$$

$$\overset{Q \ge 0}{\underset{\Lambda \ge 0}{\le}} \mathrm{tr}\, \Lambda P$$

$$\overset{\Lambda \le \mathbb{1}}{\le} \mathrm{tr}\, P = \frac{1}{2} \|\rho_0 - \rho_1\|_1$$

$$\Rightarrow \max_{0 \le \Lambda \le \mathbb{1}} \mathrm{tr}\, \Lambda(\rho_0 - \rho_1) \le \frac{1}{2} \|\rho_0 - \rho_1\|_1$$

remains to be shown: $\exists\, \Lambda$ achieving maximum.

Set $\Pi_P = \sum_{i: \lambda_i \ge 0} |i \rangle\langle i|$, projector onto support of P:

$$\Pi_P^2 = \Pi_P, \quad \Pi_P \ge 0, \quad \Pi_P \le \mathbb{1}, \quad \Pi_P P \Pi_P = P, \quad \Pi_P Q \Pi_P = 0$$

$$\mathrm{tr}\, \underline{\Pi_P} (\rho_0 - \rho_1) = \mathrm{tr}\, \Pi_P (P - Q)$$

$$= \mathrm{tr}\, \underbrace{\Pi_P P}_{= P} - \underbrace{\mathrm{tr}\, \Pi_P Q}_{= 0} = \mathrm{tr}\, P = \frac{1}{2} \|\rho_0 - \rho_1\|_1 .$$

$\square$