Recap

- ρ_{AB} is called PPT if $\rho_{AB}^{T_B} \geq 0$.

- Separability criterion: ρ_{AB} separable $\Rightarrow$ ρ_{AB} PPT

- If $|A|\cdot|B| \leq 6$, then also ρ_{AB} PPT $\Rightarrow$ ρ_{AB} SEP

- PPT channel: $(\mathrm{id}_R \otimes \mathcal{N})(\rho_{RA})$ is PPT for all ρ_{RA}

- $\mathcal{N}$ PPT $\Leftrightarrow$ Choi op $\tau^{\mathcal{N}}$ PPT $\Leftrightarrow$ $\vartheta \circ \mathcal{N}$ is CP

- Horodecki^3: PPT states are undistillable
  (cannot be converted via LOCC into maximally entangled states)

  $\Rightarrow$ PPT channels cannot generate entanglement

  $\Rightarrow$ quantum capacity $Q(\mathcal{N}) = 0$ for PPT channels

- Antidegradable channels: $\mathcal{N}: A \to B$ with comp. chan. $\mathcal{N}^c: A \to E$
  is called antidegradable, if $\mathcal{N} = \mathcal{A} \circ \mathcal{N}^c$ for some
  channel $\mathcal{A}: E \to B$ (antidegrading map).

- Antidegradable channels have $Q(\mathcal{N}) = 0$ because of no-cloning thm.

- Examples: - erasure channel $\mathcal{E}_p$ for $p \geq \frac{1}{2}$
  - amplitude damping channel $\mathcal{A}_\gamma$ for $\gamma \geq \frac{1}{2}$
  - depolarizing channel $\mathcal{D}_p$ for $p \geq \frac{1}{4}$

Antidegradability of $\mathcal{E}_p$ for $p \geq \frac{1}{2}$:

$\mathcal{H}_1 = \mathbb{C}^2$ input space

$\mathcal{H}_2 = \mathbb{C}$ erasure flag

$\mathcal{E}_p : \mathcal{B}(\mathcal{H}_1) \longrightarrow \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$

$\mathcal{E}_p : \rho \longmapsto (1-p)\tilde{\rho} + p\,\mathrm{tr}(\rho)\,|e\rangle\langle e|$

$$\hat{\rho} = \begin{pmatrix} \rho_{00} & \rho_{01} & 0 \\ \rho_{10} & \rho_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad |e\rangle\langle e| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \langle e|\tilde{\rho}|e\rangle = 0$$

$$\mathcal{E}_p^c : \mathcal{B}(\mathcal{H}_1) \to \underline{\mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)}, \quad \rho \mapsto p\tilde{\rho} + (1-p)\,\mathrm{tr}(\rho)\,|e\rangle\langle e|$$

$p \geq \frac{1}{2}$: $\quad q = \frac{2p-1}{p}$ $\quad$ Idea: erase $\tilde{\rho}$ w.p. q, do nothing with $|e\rangle\langle e|$

$$\mathcal{E}_q(p\rho) = p\overbrace{(1-q)}^{(1-p)/p}\tilde{\rho} + pq\,\mathrm{tr}(\rho)\,|e\rangle\langle e|$$

$$= \underline{(1-p)\tilde{\rho}} + (2p-1)\,|e\rangle\langle e|$$

$$\mathrm{id}\left((1-p)|e\rangle\langle e|\right) = (1-p)\,|e\rangle\langle e|$$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} + : (1-p)\tilde{\rho} + p\,|e\rangle\langle e| = \mathcal{E}_p(\rho)$$

"Problem": Need to extend the action of $\mathcal{E}_q$ to $\mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$

Solution: define A via its Kraus representation

$$\underbrace{K_0 = \sqrt{1-q}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad K_1 = \sqrt{q}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad K_2 = \sqrt{q}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{\text{erase on } \mathcal{B}(\mathcal{H}_1 \oplus 0)} \quad \underbrace{K_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{do nothing on } \mathcal{B}(0 \oplus \mathcal{H}_2)}$$

satisfies $\mathcal{E}_p = A \circ \mathcal{E}_p^c$ for $p \geq \frac{1}{2}$.

$\mathcal{B}(\mathcal{H}_1 \oplus 0)$, $\mathcal{B}(0 \oplus \mathcal{H}_2)$: $\Lambda_q = \mathcal{E}_p \oplus id$

$\mathcal{E}_q$ $\qquad$ id

2x2 block

$\downarrow$

.) for $\rho \in \mathcal{B}(\mathcal{H}_1)$, $\sigma = (1-\lambda)\tilde{\rho} + \lambda |e\rangle\langle e| = \begin{pmatrix} (1-\lambda)\rho & 0 \\ 0 & \lambda \end{pmatrix}$

$$(\mathcal{E}_q \oplus id)(\sigma) = \begin{pmatrix} \mathcal{E}_q((1-\lambda)\rho) & 0 \\ 0 & id(\lambda) \end{pmatrix}$$

.) $w \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$: $w = \begin{pmatrix} w_{\mathcal{H}_1} & * \\ * & w_{\mathcal{H}_2} \end{pmatrix}$ $\leftarrow$ off-diagonal elements

$\mathcal{H}_2 \to \mathcal{H}_1$

Can get rid of the off-diagonal blocks by measuring w.r.t.

$\mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}$.

$P_2 = |e\rangle\langle e|$, $P_1 = \mathbb{1}_{\mathcal{H}} - |e\rangle\langle e|$: $P_i \mathcal{H} = \mathcal{H}_i$

$$P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$P_1 w P_1 + P_2 w P_2 = \begin{pmatrix} w_{\mathcal{H}_1} & 0 \\ 0 & w_{\mathcal{H}_2} \end{pmatrix} \qquad w_{\mathcal{H}_i} = P_i w P_i$$

$\Lambda_M = P_1 \cdot P_1 + P_2 \cdot P_2$ $\Rightarrow$ $\Lambda = \Lambda_q \circ \Lambda_M$

when $\Lambda_q = \mathcal{E}_p \oplus id$

$\Lambda_M(\mathcal{E}_p(\rho)) = \mathcal{E}_p(\rho)$

Prop 14 A channel $\mathcal{N}: A \to B$ is antidegradable,

iff the Choi operator $\tau^{\mathcal{N}}_{AB}$ has a symmetric extension:

$\exists$ state $\sigma_{ABB'}$ s.t. i) $tr_B\, \sigma_{ABB'} = tr_{B'}\, \sigma_{ABB'} = \tau^{\mathcal{N}}_{AB}$ $\quad (B \cong B')$

ii) $\mathbb{F}_{BB'}\, \sigma_{ABB'}\, \mathbb{F}_{BB'} = \sigma_{ABB'}$

Proof: $\Rightarrow$ let $V: \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ be s.t. $\mathcal{N} = tr_E\, V . V^{\dagger}$

$|\chi\rangle_{ABE} = (\mathbb{1}_A \otimes V)\, |\gamma\rangle_{AA'}$, and

$\sigma_{ABB'} = (id_{AB} \otimes \mathcal{A})(\chi_{ABE})$, where $\mathcal{A}$ is the antideg. map

$\mathcal{N} = \mathcal{A} \circ \mathcal{N}^c$, $\mathcal{A}: E \to B \cong B'$

i) $tr_B\, \sigma_{ABB'} = tr_{B'}\, \sigma_{ABB'} = \tau^{\mathcal{N}}_{AB}$

ii) $\sigma_{ABB'}$ not necessarily symmetric, but

$$\tilde{\sigma}_{ABB'} = \frac{1}{2}\left(\sigma_{ABB'} + \mathbb{F}_{BB'}\, \sigma_{ABB'}\, \mathbb{F}_{BB'}\right)$$

is symmetric in BB', and by linearity i) still holds. ✓

$\Leftarrow$ let $\tau_{ABB'}$ be a symmetric extension of $\tau^{\mathcal{N}}_{AB}$.

let $|\psi\rangle_{ABB'R}$ be a purification of $\tau_{ABB'}$. $\quad E \cong B'R$

$\psi_{AB'R} = tr_B\, \psi_{ABB'R}$ is then the Choi operator of $\mathcal{N}^c$

$\mathcal{A} = tr_R$: $\mathcal{N} = \mathcal{A} \circ \mathcal{N}^c$ because $tr_R\, \underbrace{\psi_{AB'R}}_{= \tau^{\mathcal{N}^c}_{AE}} = \psi_{AB'} = \tau^{\mathcal{N}}_{AB}$ $\quad \square$

Corollary Let $N_1 : A \to B_1$ be antidegradable, $N_2 : B_1 \to B_2$ arbitrary.

Then $M = N_2 \circ N_1$ is also antidegradable.

<u>Proof:</u> Let $\sigma_{AB_1B_1'}$ be the symmetric extension of $\tau^{N_1}_{AB_1}$.

Then $\omega_{AB_2B_2'} = (id_A \otimes N_2 \otimes N_2)(\sigma_{AB_1B_1'})$ is a symmetric extension of $\tau^{M}_{AB_2} = (id \otimes N_2)(\tau^{N_1}_{AB_1})$.

$\overset{\text{Prop}}{\Rightarrow}$ M antidegradable. $\square$

Use this to prove that depolarizing chan. $\mathcal{D}_p$ is adg. for $p \geq \frac{1}{4}$.

$$\mathcal{D}_p : \rho \mapsto (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$$

$$\tilde{\mathcal{D}}_q : \rho \mapsto (1-q)\rho + q\, tr(\rho)\tfrac{1}{2}\mathbb{1} \qquad q = \frac{4p}{3}$$

1) $p = \frac{1}{4}$ / $q = \frac{1}{3}$: $\quad |\psi_1\rangle = \frac{1}{\sqrt{6}}(2|000\rangle + |101\rangle + |110\rangle)$

$$|\psi_2\rangle = \frac{1}{\sqrt{6}}(|001\rangle + |010\rangle + 2|111\rangle)$$

$\sigma_{ABB'} = \psi_1 + \psi_2$ is a symmetric extension of $\tau^{\tilde{\mathcal{D}}_q}_{AB}$ (check)

$\Rightarrow \tilde{\mathcal{D}}_{1/3}$ is antideg.

2) $q_1 \leq q_2 \leq 1$: $\quad \tilde{\mathcal{D}}_{q_2} = \tilde{\mathcal{D}}_w \circ \tilde{\mathcal{D}}_{q_1}$, $\quad w = \dfrac{q_2 - q_1}{1 - q_1}$ (check)

$\overset{\text{Cor}}{\Rightarrow}$ $\tilde{\mathcal{D}}_q$ is antidegradable for $\frac{1}{3} \leq q \leq 1$.

iii) $1 \leq q \leq \frac{4}{3}$ $\quad \left(\frac{3}{4} \leq p \leq 1 \right)$

Claim: $\mathcal{D}_p$ is EB for all $p \geq \frac{1}{2}$ $\quad \left(q \geq \frac{2}{3} \right)$

EB $\Leftrightarrow$ $\tau^{\mathcal{D}_p}$ is SEP $\Leftrightarrow$ $\tau^{\mathcal{D}_p}$ is PPT $\quad (|A| = |B| = 2)$

$$\tilde{\mathcal{D}}_q : \quad \tau_{AB} \equiv \tau_{AB}^{\tilde{\mathcal{D}}_q} = (1-q)\, |\gamma\rangle\langle\gamma|_{AB} + q \tfrac{1}{2}\, \mathbb{1}_2 \otimes \mathbb{1}_2$$

$$\mathbb{F}_{AB} = |\gamma\rangle\langle\gamma|^{T_B} : \quad \mathbb{F}_{AB} = \sum_{i,j} |i\rangle\langle j| \otimes |j\rangle\langle i| = \sum_{i,j} |i\rangle\langle j| \otimes |i\rangle\langle j|^T$$

$$= |\gamma\rangle\langle\gamma|^{T_B}$$

$$\Rightarrow \quad \tau_{AB}^{T_B} = \underline{\underline{(1-q)}}\, \mathbb{F}_{AB} + \frac{q}{2}\, \mathbb{1}_2 \otimes \mathbb{1}_2 \quad \overset{?}{\geq} 0$$

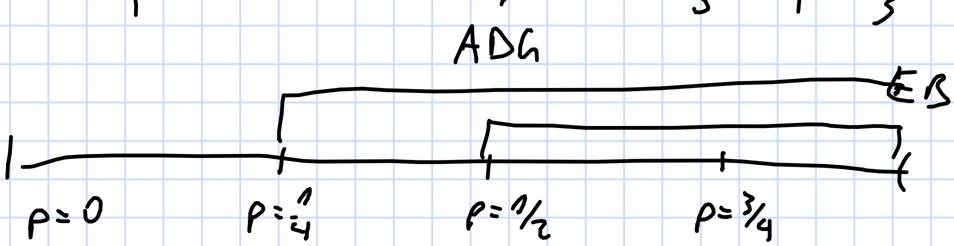
EV: $1, 1, 1, -1$ $\qquad$ EV: $1, 1, 1, 1$

$\underline{|00\rangle \quad |11\rangle \quad |01\rangle + |10\rangle}$ $\qquad$ $\underline{|01\rangle - |10\rangle}$

$q \leq 1$: $\quad \lambda_{min}\left(\tau_{AB}^{T_B}\right) = -(1-q) + \frac{q}{2} \geq 0 \Leftrightarrow q \geq \frac{2}{3}$

$q \geq 1$: $\quad \lambda_{min}\left(\tau_{AB}^{T_B}\right) = +(1-q) + \frac{q}{2} \geq 0$ ✓ if $q \leq 4/3$

$\Rightarrow$ $\tilde{\mathcal{D}}_q$ is PPT ($\Leftrightarrow$ EB) for $\frac{2}{3} \leq q \leq \frac{4}{3}$



Lemma 15 $\quad \mathcal{N}$ EB $\Rightarrow$ $\mathcal{N}$ is ADG

Proof: $\mathcal{N}$ EB $\Leftrightarrow$ $\tau_{AB}^{\mathcal{N}}$ is SEP:

$$\frac{1}{d}\tau_{AB}^{\mathcal{N}} = \sum_i p_i \, \omega_A^i \otimes \sigma_B^i$$

then $\varphi_{ABB'} = \sum_i p_i \, \omega_A^i \otimes \sigma_B^i \otimes \sigma_{B'}^i$ is a symmetric

extension of $\frac{1}{d}\tau_{AB}$ $\Rightarrow$ $\mathcal{N}$ is ADG. $\square$

Our extension is usually called 2-extension.

More generally: ρ_{AB} is called k-extendible, if $\exists \, \sigma_{AB_1\ldots B_k}$

s.t. $\quad$ i) $\mathrm{tr}_{B_i^c}\, \sigma_{AB_1\ldots B_k} = \rho_{AB}$

ii) $P_\pi \, \sigma_{AB_1\ldots B_k} P_\pi^\dagger = \sigma_{AB_1\ldots B_k} \quad \forall \pi \in S_k$

·) Separable states are ∞-extendible

·) ρ_{AB} k-extendible $\Rightarrow$ ρ_{AB} ℓ-extendible for $\ell \leq k$.