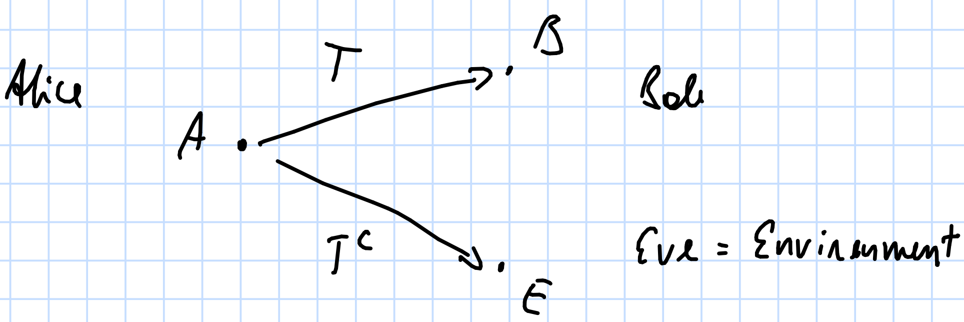
Recap

·) Isometric picture of quantum channel $T: A \to B$

$$V: \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E \text{ isometry}, \quad T(X_A) = \mathrm{tr}_E\, V X V^\dagger$$

·) Complementary channel $T^c: A \to E$, $T^c(X_A) = \mathrm{tr}_B\, V X V^\dagger$

·) T^c models loss of information to environment



·) Stinespring isometry V is not unique: $\mathrm{tr}_E\, V X V^\dagger = \mathrm{tr}_E\, \tilde{V} X \tilde{V}^\dagger$

for any $\tilde{V} = (\mathbb{1} \otimes U_E) V$ with U_E unitary

·) Leads to family of complementary channels: $\{ U_E T^c(.) U_E^\dagger \}$

·) Some important qubit channels:

-) Flip or dephasing channels $\mathcal{F}_p^\theta: \rho \mapsto (1-p)\rho + p\, \theta \rho \theta^\dagger$

where $\theta \in \{X, Y, Z\}$ Pauli operators

-) Depolarizing channel: $\mathcal{D}_p: \rho \mapsto (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$

alt.: $\mathcal{D}_q: \rho \mapsto (1-q)\rho + q\, \mathrm{tr}(\rho)\, \frac{1}{2}\mathbb{1} \qquad (q = \frac{4p}{3})$

-) generalized Pauli chan.: $N_{\underline{p}}(\rho) = p_0 \rho + p_1 X\rho X + p_2 Y\rho Y + p_3 Z\rho Z$

$\underline{p} = (p_0, p_1, p_2, p_3)$ prob. dist

·) mixed unitary channels: $\{U_i\}_{i=1}^{k}$ unitaries, $p=(p_1,\dots,p_k)$

$$\mathcal{N}(\rho)=\sum_{i=1}^{k} p_i\, U_i \rho U_i^\dagger$$

$$\Rightarrow \mathcal{N}(\mathbb{1})=\sum_{i=1}^{k} p_i \underbrace{U_i U_i^\dagger}_{=\mathbb{1}} = \sum_i p_i\, \mathbb{1} = \underline{\mathbb{1}}$$

·) Q: In what sense is a flip channel dephasing?

$$\mathcal{F}_p^Z : \rho \mapsto (1-p)\rho + p\, Z\rho Z$$

dephase off-diagonal elements

$$\rho=\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix} : \quad \mathcal{F}_p^Z(\rho)=\begin{pmatrix}\rho_{11} & (1-2p)\rho_{12}\\ (1-2p)\rho_{21} & \rho_{22}\end{pmatrix}$$

$\rho \geq 0$, $tr(\rho)=1$

Observations: a) If $\rho = x\,|0\rangle\langle 0| + (1-x)\,|1\rangle\langle 1|$

$\Rightarrow \mathcal{F}_p^Z(\rho)=\rho \quad \forall p\in[0,1]$

b) $p=\frac{1}{2}$: $\mathcal{F}_{1/2}^Z$ is diagonal in Z-basis for all ρ

Let's send classical information through $\mathcal{F}_p^Z$!

$|0\rangle\langle 0|$, $|1\rangle\langle 1|$: encode 1 classical bit in a qubit

$\uparrow$ $\uparrow$

0 1

classical bits

$$\mathcal{F}_p^Z(|0\rangle\langle 0|)=|0\rangle\langle 0|, \quad \mathcal{F}_p^Z(|1\rangle\langle 1|)=|1\rangle\langle 1|$$

$\Rightarrow$ can send 1 bit through $\mathcal{F}_p^Z$!

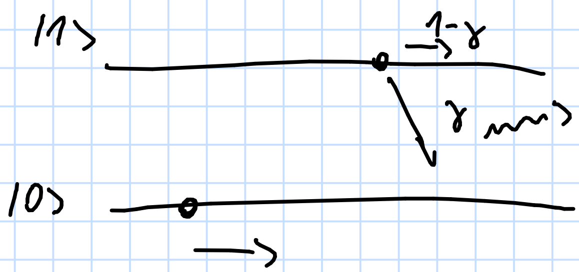
This is optimal: can send at most 1 bit of classical info through qubit channel.

f) Amplitude-damping channel

physical model: 2-level system, e.g. an atom with

ground state $|0\rangle$, excited state $|1\rangle$

If system is in excited state $|1\rangle$, it decays with certain probability γ, emitting a photon to the environment.



Isometry: $|0\rangle_A \mapsto |0\rangle_B \otimes |0\rangle_E$

$|1\rangle_A \mapsto \sqrt{1-\gamma}\, |1\rangle_B |0\rangle_E + \sqrt{\gamma}\, |0\rangle_B |1\rangle_E$

compactly: $V = |0\rangle_B |0\rangle_E \langle 0|_A + \left(\sqrt{1-\gamma}\, |1\rangle_B |0\rangle_E + \sqrt{\gamma}\, |0\rangle_B |1\rangle_E\right) \langle 1|_A$

Kraus op's: $K_0 = \langle 0|_E V = |0\rangle\langle 0|_{BA} + \sqrt{1-\gamma}\, |1\rangle\langle 1|_{BA} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}$

$K_1 = \langle 1|_E V = \sqrt{\gamma}\, |0\rangle_B \langle 1|_A = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$

Amplitude damping channel $\mathcal{A}_\gamma : \rho \mapsto K_0 \rho K_0^\dagger + K_1 \rho K_1^\dagger$

·) not unital: $\mathcal{A}_p(\mathbb{1}) = \begin{pmatrix} 1+\gamma & 0 \\ 0 & 1-\gamma \end{pmatrix} \neq \mathbb{1}$ for $\gamma \neq 0$

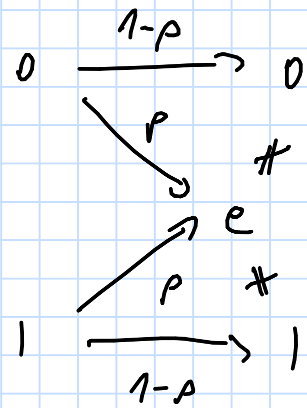
·) $\mathcal{A}_p$ is not a Pauli channel (it's not even a mixed unitary channel)

·) Completely understood how to send quantum info through $\mathcal{A}_p$.

Not understood how much classical info can be sent through $\mathcal{A}_p$.

g) erasure channel

classical model:



$\{\mathbb{1}, |e\rangle\langle e|\}$

$|e\rangle \perp \rho \in \mathcal{B}(\mathcal{H})$

$\Leftrightarrow \langle e|\rho|e\rangle = 0 \quad \forall \rho$

quantum version: $\mathcal{E}_p : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H} \oplus \mathbb{C})$

$\mathcal{H} \cong \mathbb{C}^2$

$$\rho \mapsto (1-p)\rho + p\,\mathrm{tr}(\rho)\,|e\rangle\langle e|$$

$\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 0 \end{pmatrix}$

Kraus operators: $K_0 = \sqrt{1-p}\,\mathbb{1}$, $\quad K_1 = \sqrt{p}\,|e\rangle\langle 0|$, $\quad K_2 = \sqrt{p}\,|e\rangle\langle 1|$

Bob can always tell whether erasure happened by performing a measurement!

§ 2.2. Generalized dephasing channels

A gen. dephasing channel leaves a fixed ONB invariant and dephases off-diagonal elements w.r.t. that basis.

$$X = \begin{pmatrix} x_{11} & & * \\ & x_{22} & \\ & & \ddots & \\ * & & & x_{dd} \end{pmatrix} \xmapsto[\text{dephasing}]{\text{gen.}} \begin{pmatrix} x_{11} & & \\ & x_{22} & \\ & & \ddots & \\ & & & x_{dd} \end{pmatrix}$$

dephasing

dephasing

Construction: $\mathcal{H} = \mathbb{C}^d$, $\{|i\rangle\}_{i=1}^d$ ONB for $\mathcal{H}$

Choose environment $\mathcal{H}_E$ with $|E| \geq 2$, and let

$\{|\varphi_i\rangle_E\}_{i=1}^d$ be <u>some</u> set of pure states on E.

$\langle\varphi_i|\varphi_j\rangle \neq \delta_{ij}$ $\quad (\langle\varphi_i|\varphi_i\rangle = 1 \ \forall i)$

Isometry V: $\underline{|i\rangle_A \mapsto |i\rangle_B \otimes |\varphi_i\rangle_E}$

$$\mathcal{N}(\rho_A) = \operatorname{tr}_E V\rho_A V^\dagger = \sum_{i,j} \langle i|\rho|j\rangle\, |i\rangle\langle j|_B \operatorname{tr}(|\varphi_i\rangle\langle\varphi_j|)$$

$$= \sum_{i,j} \langle\varphi_i|\varphi_j\rangle \langle i|\rho|j\rangle\, |i\rangle\langle j|_B$$

$[\mathcal{N}(\rho)]_{kk} = \langle\varphi_k|\varphi_k\rangle\langle k|\rho|k\rangle = \rho_{kk}$ ✓

$[\mathcal{N}(\rho)]_{jk} = \langle j|\rho|k\rangle \overbrace{\langle\varphi_j|\varphi_k\rangle}^{\neq 0}$ for $j \neq k$.

Examples: $\mathcal{F}_p^z$, $\mathcal{F}_p^y$, $\mathcal{F}_p^x$ (Ex.: what are the $|\varphi_i\rangle$'s for these channels?)

Higher-dim. example: Let $d \geq 2$, and define two unitaries

$X|i\rangle = |i+1 \bmod d\rangle$ $\qquad$ $Z|j\rangle = \omega^j|j\rangle$

"shift operator" $\qquad$ "clock operator" where $\omega = \exp(\frac{2\pi i}{d})$

$\to$ generalize Pauli operators: $X^d = Z^d = \mathbb{1}$

generators of Heisenberg-Weyl group: $\{\omega^j Z^k X^l : j,k,l \in [d]\}$

As matrices:

$$X = \begin{pmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & 1 & \ddots & \\ & & 1 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{d-1} \end{pmatrix}$$

$\varrho \in \mathcal{B}(\mathbb{C}^d)$:

$$\left.\begin{array}{l} \varrho \mapsto (1-p)\,\varrho + p\, X \varrho X^\dagger \\[2ex] \varrho \mapsto (1-p)\,\varrho + \frac{p}{3}\, Z \varrho Z^\dagger + \frac{2p}{3}\, Z^2 \varrho (Z^2)^\dagger \end{array}\right\} \text{generalized dephasing channels}$$