Recap

- Kraus representation of quantum channel $T: \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$

$$T(X) = \sum_i K_i X K_i^\dagger \quad \text{with} \quad \sum_i K_i^\dagger K_i = \mathbb{1}_1$$

- Kraus rank $r(T) = \min \#$ Kraus op's, equal to rank of Choi op.
- NOT unique: Let U be unitary, then $L_i := \sum_j U_{ij} K_j$

  is another Kraus representation.

Quantum channel $T: \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$

Kraus representation	Isometric picture	Unitary picture
$T(X) = \sum_i K_i X K_i^\dagger$	$T(X) = \mathrm{tr}_E\, V X V^\dagger$	$T(X) = \mathrm{tr}_E\, U(X \otimes \varphi) U^\dagger$
$\sum_i K_i^\dagger K_i = \mathbb{1}_1$	$V: \mathcal{H}_1 \to \mathcal{H}_2 \otimes \mathcal{H}_E$	U unitary
	Isometry: $V^\dagger V = \mathbb{1}_1$	$\varphi = \lvert\varphi\rangle\langle\varphi\rvert$ fixed state

Kraus representation $\xrightarrow{V = \sum_i K_i \otimes \lvert i\rangle}$ Isometric picture $\xrightarrow{\mathcal{H}_E = \mathcal{H}_1 \otimes \mathcal{H}_2,\ \text{complete unitary}}$ Unitary picture

Kraus representation $\xleftarrow{K_i = (\mathbb{1}_1 \otimes \langle i\rvert) V}$ Isometric picture $\xleftarrow{V = U(\mathbb{1}_1 \otimes \lvert\varphi\rangle)}$ Unitary picture

- Important fact #1: If $\sum_i \lvert\psi_i\rangle\langle\psi_i\rvert = \sum_j \lvert\varphi_j\rangle\langle\varphi_j\rvert$,

  then $\exists\, U$ unitary s.t. $\lvert\psi_i\rangle = \sum_j U_{ij} \lvert\varphi_j\rangle$

- Important fact #2: Any two purifications of a state ρ are related

  by an isometry acting on the purifying system (Ex)

§ 1.6 Linear representation

$T: B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ linear map

Linear algebra: linear maps on finite-dim. Hilbert spaces correspond to matrices (w.r.t. fixed basis)

·) $B(\mathcal{H})$ ... Hilbert space with inner product $\langle X, Y\rangle = \mathrm{tr}(X^\dagger Y)$

·) treat op's $X \in B(\mathcal{H})$ as vectors, and maps $T: B(\mathcal{H}) \to B(\mathcal{H})$ as matrices.

·) Let $\{|i\rangle\}_{i=1}^{d}$, $d = \dim \mathcal{H}$, be an ONB for $\mathcal{H}$.

Then $\{|i\rangle\langle j|\}_{i,j=1}^{d}$ is a basis for $B(\mathcal{H}) \Rightarrow \dim B(\mathcal{H}) = d^2$.

·) We define a linear mapping

$$\mathrm{vec}: B(\mathcal{H}) \to \mathcal{H} \otimes \mathcal{H}$$

$$|i\rangle\langle j| \mapsto |i\rangle \otimes |j\rangle \text{ + linear extension}$$

·) $\{|i\rangle \otimes |j\rangle\}_{i,j=1}^{d}$ is a basis for $\mathcal{H} \otimes \mathcal{H} \Rightarrow$ vec is an isomorphism $B(\mathcal{H}) \cong \mathcal{H} \otimes \mathcal{H}$

Properties of vec:

·) $\mathrm{vec}(|\psi\rangle\langle\varphi|) = |\psi\rangle \otimes |\bar{\varphi}\rangle$ ($\bar{\ }$ denotes complex conjugation w.r.t. the basis $\{|i\rangle\}_i$)

·) vec is an isometry:

$$\langle X, Y\rangle_{B(\mathcal{H})} = \langle \mathrm{vec}(X), \mathrm{vec}(Y)\rangle_{\mathcal{H}\otimes\mathcal{H}} \leftarrow \text{inner product induced by } \langle\cdot,\cdot\rangle \text{ on } \mathcal{H}$$

·) $\mathrm{vec}(AXB) = (A \otimes B^T)\,\mathrm{vec}(X)$

Linear map $\mathcal{N} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \quad \overset{vec}{\longleftrightarrow} \quad$ operator $N \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$

$$\mathcal{N}(X) = \sum_i A_i X B_i^T \quad \overset{vec}{\longleftrightarrow} \quad N = \sum_i A_i \otimes B_i$$

$$\left(\text{using identity } vec(AXB^T) = (A \otimes B)\, vec(X) \right)$$

CP maps: Kraus representation

$$\mathcal{N}(X) = \sum_i K_i X K_i^\dagger \quad \overset{vec}{\longleftrightarrow} \quad N = \sum_i K_i \otimes \bar{K}_i$$

transfer matrix of channel $\mathcal{N}$

Unitality: $\sum_i K_i K_i^\dagger = \mathbb{1} \quad \overset{vec}{\longleftrightarrow} \quad N|\gamma\rangle = \left(\sum_i K_i \otimes \bar{K}_i\right)|\gamma\rangle = |\gamma\rangle = vec(\mathbb{1})$

trace-preserving: $\sum_i K_i^\dagger K_i = \mathbb{1} \quad \overset{vec}{\longleftrightarrow} \quad N^\dagger|\gamma\rangle = \left(\sum_i K_i^\dagger \otimes K_i^T\right)|\gamma\rangle = |\gamma\rangle$

[If $T = \sum_i A_i \cdot B_i^T$, then $T^\dagger = \sum_i B_i^T \cdot A_i$]

The linear representation is useful, because it translates channel composition into matrix multiplication:

$\boxed{\text{Prop 7}}$ Let $\mathcal{M} : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$, $\mathcal{N} : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_3)$ be linear maps with transfer matrices M, N.

Then $N \cdot M$ is the transfer matrix of $\mathcal{N} \circ \mathcal{M} : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_3)$.

<u>Proof</u>: $\mathcal{M} = \sum_i A_i X B_i^T \qquad A_i, B_i \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$

$\mathcal{N} = \sum_i C_i X D_i^T \qquad C_i, D_i \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$

$$\mathcal{N} \circ \mathcal{M} = \sum_{i,j} C_i A_j X \underbrace{B_j^T D_i^T}_{(D_i B_j)^T}$$

$\Big\updownarrow vec$

$$M = \sum_i A_i \otimes B_i \,, \quad N = \sum_i C_i \otimes D_i \quad \Rightarrow \quad N \cdot M = \sum_{i,j} C_i A_j \otimes D_i B_j \qquad \square$$

$\boxed{\text{Prop 8}}$ Let $\mathcal{N}: \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ be a linear map with transfer matrix N and Choi operator $\tilde{\tau} = (\mathcal{N} \otimes id_1)(\gamma)$.

Then $\tilde{\tau} = N^\Gamma$, where Γ is an involution ($\Gamma^2 = id$) defined by $\langle i,j | X^\Gamma | k,l \rangle = \langle i,k | X | j,l \rangle$. ($|i,j\rangle \equiv |i\rangle \otimes |j\rangle$)

__Proof:__ w.l.o.g. $\mathcal{N} = X \cdot Y^T$, where $X, Y \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$

ONB $\{|\alpha\rangle\}_\alpha$

ONB $\{|i\rangle\}_i$

$$X = \sum_{i,\alpha} x_{\alpha,i} |\alpha\rangle\langle i| \quad , \quad Y = \sum_{j,\beta} y_{\beta,j} |\beta\rangle\langle j|$$

$$N = X \otimes Y = \sum_{\substack{i,j \\ \alpha,\beta}} x_{\alpha,i}\, y_{\beta,j}\, |\alpha\rangle\langle i| \otimes |\beta\rangle\langle j|$$

$$\tilde{\tau} = (\mathcal{N} \otimes id_2)(\gamma) = \sum_{k,l} X |k\rangle\langle l| Y^T \otimes |k\rangle\langle l|$$

$$= \sum_{\substack{k,l \\ i,j,\alpha,\beta}} x_{\alpha,i}\, y_{\beta,j}\, |\alpha\rangle \underbrace{\langle i|k\rangle}_{\delta_{ik}} \underbrace{\langle l|j\rangle}_{\delta_{lj}} \langle\beta| \otimes |k\rangle\langle l|$$

$$= \sum x_{\alpha,i}\, y_{\beta,j}\, |\alpha\rangle\langle\beta| \otimes |i\rangle\langle j|$$

$$\langle \alpha,\beta | N | i,j \rangle = \langle \alpha, i | \tilde{\tau} | \beta, j \rangle \qquad \square$$

Note that $\tilde{\tau} \in \mathcal{B}(\mathcal{H}_2 \otimes \mathcal{H}_1, \mathcal{H}_2 \otimes \mathcal{H}_1)$ Γ

while $N \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_1, \mathcal{H}_2 \otimes \mathcal{H}_2)$