Recap

Choi operator $\tau \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$

·) Choi-Jamiołkowski isomorphism: linear map $T: B(\mathcal{H}_1) \to B(\mathcal{H}_2)$

$$T \mapsto \tau = (id_1 \otimes T)(\gamma), \qquad \tau \mapsto \left[T(X) = tr_1\left(\tau\left(X^T \otimes \mathbb{1}_2\right)\right)\right]$$

·) $|\gamma\rangle = \sum_{i=1}^{\dim \mathcal{H}_1} |i\rangle \otimes |i\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_1$ unnormalized maximally entangled state

·) Choi operator encodes properties of linear map T

·) T quantum channel: T is CP $\iff$ $\tau \geq 0$

T is TP $\iff$ $tr_2 \tau = \mathbb{1}_1$

·) unitality: $tr_1 \tau = \mathbb{1}_1$

·) Examples of CP maps:

- partial trace $X_{12} \mapsto tr_2 X_{12}$ (also TP)
- isometries $V: \mathcal{H}_1 \to \mathcal{H}_2$ s.t. $V^\dagger V = \mathbb{1}_1$ (also TP)
- maps $X \mapsto \sum_i K_i X K_i^\dagger$ for arbitrary op's $\{K_i\}_i$

·) positive but not completely positive:

transposition $\vartheta: X \mapsto X^T$

Choi operator $\mathbb{F} = (id \otimes \vartheta)(\gamma)$ is the swap operator, $\mathbb{F} \not\geq 0$.

·) Kraus representation: <u>every</u> CP map can be written in the form

$$X \mapsto \sum_i K_i X K_i^\dagger$$

Prop 5 a) A map $T: \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ is CP iff $\exists \{K_i\}_i$ with $K_i \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ s.t. $T(X) = \sum_i K_i X K_i^\dagger$.

K_i ... Kraus operators of T.

b) The Kraus rank $r(T)$, the minimal number of Kraus op's, is equal to the rank of the Choi op. $\tau = (id \otimes T)(\gamma)$.

$r(T) \leq \dim \mathcal{H}_1 \dim \mathcal{H}_2$

c) There exists a Kraus representation with $r = \text{rank}(\tau)$ op's s.t. $\langle K_i, K_j \rangle = tr(K_i^\dagger K_j) = \delta_{ij} c_i$.

d) T is TP iff $\sum_i K_i^\dagger K_i = \mathbb{1}_1$, T is unital iff $\sum_i K_i K_i^\dagger = \mathbb{1}_2$.

e) Any two Kraus rep's $\{K_i\}_i$, $\{L_i\}_i$ of a channel T are related by a unitary U: $K_i = \sum_{j} U_{ij} L_j$

$$\left(T(X) = \sum_i K_i X K_i^\dagger = \sum_j L_j X L_j^\dagger \right)$$

__Proof__: a) T is CP iff $T(X) = \sum_i K_i X K_i^\dagger$ for some op's $\{K_i\}$.

($\Leftarrow$) ✓

($\Rightarrow$) Prop 4: T is CP iff $\tau = (id \otimes T)(\gamma) \geq 0$.

__Claim__: $X \geq 0 \iff \exists \{|\psi_i\rangle\}_{i=1}^r$ with $r \geq \text{rk}\, X$ s.t. $X = \sum_i |\psi_i\rangle\langle\psi_i|$. $\langle\psi_i|\psi_j\rangle \neq \delta_{ij}$ in general

($\Leftarrow$) $X = \sum_i |\psi_i\rangle\langle\psi_i|$ for some $\{|\psi_i\rangle\}_i$.

$X \geq 0 \;:\Leftrightarrow\; \langle\varphi|X|\varphi\rangle \geq 0 \;\; \forall |\varphi\rangle \;\rightarrow\; \langle\varphi|\left(\sum_i |\psi_i\rangle\langle\psi_i|\right)|\varphi\rangle$

$= \sum_i |\langle\varphi|\psi_i\rangle|^2 \geq 0 \;\checkmark$

($\Rightarrow$) $X \geq 0$: $\quad X = \sum_i \lambda_i |\chi_i\rangle\langle\chi_i|$ where $X|\chi_i\rangle = \lambda_i|\chi_i\rangle$

$\langle\chi_i|\chi_j\rangle = \delta_{ij}$

$|\psi_i\rangle := \sqrt{\lambda_i}\,|\chi_i\rangle \;\Rightarrow\; X = \sum_i |\psi_i\rangle\langle\psi_i|$. "$\square$"

$\tau \geq 0 \overset{\text{Claim}}{\Rightarrow} \exists \{|\psi_i\rangle\}$ s.t. $\tau = \sum_i |\psi_i\rangle\langle\psi_i|$

$\forall i: \;\exists K_i$ s.t. $|\psi_i\rangle = (\mathbb{1}_1 \otimes K_i)|\gamma\rangle$

$\tau = \sum_i |\psi_i\rangle\langle\psi_i| = \sum_i (\mathbb{1}_1 \otimes K_i)|\gamma\rangle\langle\gamma|(\mathbb{1}_1 \otimes K_i)^\dagger$

C-J isomorphism $\leftrightarrow T(X) = \sum_i K_i X K_i^\dagger$.

b) Kraus rank $r(T) = \mathrm{rk}(\tau) \Rightarrow$ clear from a)

c) Let $\tau = \sum_i \lambda_i |\chi_i\rangle\langle\chi_i|$ be the spectral decomposition.

with $|\varphi_i\rangle = \sqrt{\lambda_i}\,|\chi_i\rangle$: $\tau = \sum_i |\varphi_i\rangle\langle\varphi_i| = \sum_i (\mathbb{1}\otimes L_i)|\gamma\rangle\langle\gamma|(\mathbb{1}\otimes L_i)^\dagger$

where $|\varphi_i\rangle = (\mathbb{1}\otimes L_i)|\gamma\rangle$

Then:

$\lambda_i \delta_{ij} = \langle\varphi_i|\varphi_j\rangle = \langle\gamma|(\mathbb{1}\otimes L_i)^\dagger(\mathbb{1}\otimes L_j)|\gamma\rangle = \sum_{k,l} \underbrace{\langle k|l\rangle}_{=\delta_{kl}} \langle k|L_i^\dagger L_j|l\rangle$

$= \sum_k \langle k|L_i^\dagger L_j|k\rangle = \mathrm{tr}(L_i^\dagger L_j) = \langle L_i, L_j\rangle$.

d) T is TP iff $\sum_i K_i^\dagger K_i = \mathbb{1}_1$ and T is unital iff $\sum_i K_i K_i^\dagger = \mathbb{1}_2$

$\cdot)$ $\operatorname{tr}(T(X)) = \sum_i \operatorname{tr}(K_i X K_i^\dagger) = \sum_i \operatorname{tr}(K_i^\dagger K_i X)$

$$= \operatorname{tr}\Big(\sum_i K_i^\dagger K_i X\Big) = \operatorname{tr} X \quad \forall X$$

$$\Leftrightarrow \sum_i K_i^\dagger K_i = \mathbb{1}_1.$$

$\cdot)$ $T(\mathbb{1}_1) = \sum_i K_i \mathbb{1} K_i^\dagger = \sum_i K_i K_i^\dagger = \mathbb{1}_2$.

e) $T = \sum_i K_i \cdot K_i^\dagger = \sum_j L_j \cdot L_j^\dagger \Leftrightarrow \exists U$ s.t. $K_i = \sum_j U_{ij} L_j$

<u>Claim:</u> $\underline{\sum_i |\psi_i\rangle\langle\psi_i|} = \underline{\sum_j |\varphi_j\rangle\langle\varphi_j|}$ iff $\exists U$ with $|\psi_i\rangle = \sum_j U_{ij} |\varphi_j\rangle$

<u>Proof:</u> Consider purifications

$|\Psi\rangle = \sum_i |\psi_i\rangle \otimes |i\rangle$ and $|\Phi\rangle = \sum_j |\varphi_j\rangle \otimes |j\rangle$

reference system with ONB $\{|i\rangle\}_i$

<u>there exists an isometry V s.t. $|\Psi\rangle = (\mathbb{1} \otimes V)|\Phi\rangle$</u>

$\rightarrow$ Isometry V can be extended to a unitary U: $|\Psi\rangle = (\mathbb{1} \otimes U)|\Phi\rangle$

$$|\psi_i\rangle = \underline{(\mathbb{1} \otimes \langle i|)|\Psi\rangle} = (\mathbb{1} \otimes \langle i|(\mathbb{1} \otimes U)|\Phi\rangle$$

$$= (\mathbb{1} \otimes \langle i|(\mathbb{1} \otimes U)\sum_j |\phi_j\rangle \otimes |j\rangle$$

$$= \sum_j |\phi_j\rangle \overbrace{\langle i|V|j\rangle}^{U_{ij}} \qquad \square$$

Isometry: $V : \mathcal{H}_1 \to \mathcal{H}_2$: $\langle \psi | \varphi \rangle = \langle \psi | V^\dagger V | \varphi \rangle \ \forall |\psi\rangle, |\varphi\rangle \in \mathcal{H}_1$

$$\Leftrightarrow \underline{V^\dagger V = \mathbb{1}_1.}$$

$\dim \mathcal{H}_2 \geq \dim \mathcal{H}_1$. If $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 \Rightarrow V^\dagger V = V V^\dagger = \mathbb{1}$ so V is unitary.

often for us: $\mathcal{H}_2 = \mathcal{H}_B \otimes \mathcal{H}_E$

$\boxed{\text{Prop 6}}$ a) T is CP iff $\exists\, V : \mathcal{H}_1 \to \mathcal{H}_2 \otimes \mathbb{C}^r$

$T : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$

- $r \geq r(T)$
- $T(X) = \mathrm{tr}_E\, \underline{V X V^\dagger}$

environment $= \mathbb{C}^r$.

b) T is TP iff $V : \mathcal{H}_1 \to \mathcal{H}_2 \otimes \mathbb{C}^r$ is an isometry.

Proof: a) $\Leftarrow$ ✓

$\Rightarrow$ T is CP: $\exists \{K_i\}_{i=1}^r$ s.t. $T(X) = \sum_i K_i X K_i^\dagger$, $r \geq r(T)$

choose ONB $\{|i\rangle\}_{i=1}^r$ in $\mathbb{C}^r$: $V = \sum_{i=1}^r K_i \otimes |i\rangle$

$$\mathrm{tr}_E\, V X V^\dagger = \sum_{i,j} \mathrm{tr}_1 \left((K_i \otimes |i\rangle) X (K_j \otimes |j\rangle)^\dagger \right)$$

$$= \sum_{i,j} K_i X K_j^\dagger \underbrace{\mathrm{tr}(|i\rangle\langle j|)}_{\delta_{ij}} = \sum_i K_i X K_i^\dagger = T(X)$$

b) T is TP iff $V^\dagger V = \mathbb{1}$:

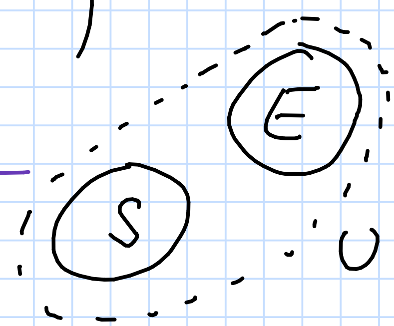
$$\operatorname{tr} X = \operatorname{tr} T(X) = \operatorname{tr}\left(\operatorname{tr}_E VXV^\dagger\right) = \operatorname{tr}(VXV^\dagger)$$
$$= \operatorname{tr}(V^\dagger V X) = \operatorname{tr} X \quad \forall X \iff V^\dagger V = \mathbb{1} \qquad \square$$

·) The isometry V in Prop 6 is called Stinespring isometry or Stinespring dilation of the channel T.

·) Given Stinespring isometry V, a Kraus rep. $\{K_i\}_i$ is obtained via $K_i = (\mathbb{1} \otimes \langle i|)V \longleftrightarrow T(X) = \operatorname{tr}_E VXV^\dagger$

(remember: $\operatorname{tr}_E Y = \sum_i (\mathbb{1} \otimes \langle i|) Y (\mathbb{1} \otimes |i\rangle)$)

§ 1.5 Unitary picture and open system dynamics



Quantum channel $T: \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$:

$\exists$ Stinespring isom. $V: \mathcal{H}_1 \to \mathcal{H}_2 \otimes \underline{\mathbb{C}^{r}}$ s.t. $T(X) = \operatorname{tr}_E VXV^\dagger$

Choose: $r = d_1 \cdot d_2$, $d_i = \dim \mathcal{H}_i$

Complete V to unitary U acting on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_2}$ such that

$$V = U(\mathbb{1}_{d_1} \otimes |\varphi\rangle) \quad \text{and} \quad T(X) = \operatorname{tr}_E VXV^\dagger$$

some fixed vector in $\mathbb{C}^{d_2} \otimes \mathbb{C}^{d_2}$

$$= \operatorname{tr}_{E'} U(X \otimes |\varphi\rangle\langle\varphi|)U^\dagger$$

$$E' \cong \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$$