Lecture 40: Determinant formulas

Last time: Existence and uniqueness of the determinant

Determinant: function $M_n(\mathbb{F}) \to \mathbb{F}$ satisfying

- alternating
- multilinear
- normalized

Crucial step in the proof: Let $A \in M_n(\mathbb{F})$ and for fixed $i, j \in \{1, \dots, n\}$

let $\tilde{A}_{ij} \in M_{n-1}(\mathbb{F})$ be the matrix obtained from removing the

i-th row and j-th column of A.

For fixed $i \in \{1, \dots, n\}$, we have the Laplace expansion

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(\tilde{A}_{ij})$$

Explicit formulas for det:

$n=1$: $M_1(\mathbb{F}) \cong \mathbb{F}$. Let $x \in \mathbb{F}$, then $x = x \cdot 1$,

$$\text{and } \det(x) = \det(x \cdot 1) = x \underbrace{\det(1)}_{=1} = x$$

$n=2$: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{F})$ and fix $i=1$:

$$\tilde{A}_{11} = d, \quad \tilde{A}_{12} = c \Rightarrow \det(A) = (-1)^{1+1} a_{11} \det(\tilde{A}_{11}) + (-1)^{1+2} a_{12} \det(\tilde{A}_{12})$$

$$= a \cdot d - b \cdot c$$

$\underline{n=3:}$ let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in M_3(\mathbb{F})$

$i=1:$

$$\det(A) = (-1)^{1+1} a_{11} \underbrace{\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}}_{\longrightarrow\, a_{22}a_{33} - a_{23}a_{32}}$$

$$+ (-1)^{1+2} a_{12} \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$$

$$+ (-1)^{1+3} a_{13} \det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$$= \underline{a_{11}a_{22}a_{33}} + \underline{a_{12}a_{23}a_{31}} + \underline{a_{13}a_{21}a_{32}}$$

$$- \underline{a_{11}a_{23}a_{32}} - \underline{a_{12}a_{21}a_{33}} - \underline{a_{13}a_{22}a_{31}}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

General closed formula for arbitrary n:

Def A permutation of $\{1,\dots,n\}$ is a bijection $=\{1,\dots,n\}$

$\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$, that is, $\{\sigma(1),\sigma(2),\dots,\sigma(n)\}$

The set of all permutations forms a group called the symmetric group, denoted S_n. We have $|S_n| = n!$

For $\sigma \in S_n$, we define a so-called permutation matrix $P_\sigma \in M_n(\mathbb{F})$:

$$(P_\sigma)_{ij} = \begin{cases} 1 & \text{if } j = \sigma(i) \\ 0 & \text{otherwise} \end{cases}$$

E.g., $n=4$ $\quad \left.\begin{array}{l} \sigma: 1 \to 2 \\ \quad\ 2 \to 3 \\ \quad\ 3 \to 1 \\ \quad\ 4 \to 4 \end{array}\right\} \quad P_\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\boxed{\text{Def}}$ The sign of a permutation $\sigma \in S_n$, denoted $\operatorname{sign}(\sigma)$, is defined as $\operatorname{sign}(\sigma) = \det(P_\sigma)$.

$\operatorname{sign}(\sigma) = (-1)^k$ where k is the number of swaps or transpositions needed to bring $(\sigma(1), \dots, \sigma(n))$ to $(1, \dots, n)$.

$\boxed{\text{Prop}}$ Determinant formula

$A \in M_n(\mathbb{F})$: $\quad \det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{\sigma(1),1} \cdot a_{\sigma(2),2} \cdot \ldots \cdot a_{\sigma(n),n}$

Ex.: $n=3$:

permutations	sign
1 2 3	+1
1 3 2	−1
2 1 3	−1
2 3 1	+1
3 1 2	+1
3 2 1	−1

$A \in M_3(\mathbb{F})$

$$\det(A) = a_{11} a_{22} a_{33} + a_{21} a_{32} a_{13} + a_{31} a_{12} a_{23} - a_{11} a_{32} a_{23} - a_{21} a_{12} a_{33} - a_{31} a_{22} a_{13}$$

Proof: Write $A = (\alpha_1 | \dots | \alpha_n)$, $\alpha_j \in \mathbb{F}^n$, $\alpha_j = \sum_{i_j=1}^{n} a_{i_j j} e_{i_j}$

by multilinearity,

$$\det(A) = \det\left(\sum_{i_1} a_{i_1 1} e_{i_1} \Big| \dots \Big| \sum_{i_j} a_{i_j j} e_{i_j} \Big| \dots \Big| \sum_{i_n} a_{i_n n} e_{i_n} \right)$$

$$= \sum_{i_1} \dots \sum_{i_j} \dots \sum_{i_n} a_{i_1 1} \dots a_{i_j j} \dots a_{i_n n} \cdot \underbrace{\det\left(e_{i_1} | \dots | e_{i_j} | \dots | e_{i_n} \right)}_{=(*)}$$

Since det is alternating,

$(*)$ is zero unless $\{i_1, \dots, i_n\} = \{1, \dots, n\}$, or in other words,

$\sigma = (i_1, \dots, i_n)$ is a permutation in S_n.

But $(*)$ is a matrix P with elements $P_{ij} = \begin{cases} 1 & \text{if } i = \sigma(j) \\ 0 & \text{otherwise} \end{cases}$

$i = \sigma(j) \Leftrightarrow j = \sigma^{-1}(i)$, and hence $P = P_{\sigma^{-1}}$,

and $\det(P_{\sigma^{-1}}) = \det(P_\sigma) = \operatorname{sign}(\sigma)$.

$$\Rightarrow \det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{\sigma(1),1} \cdot \ldots \cdot a_{\sigma(j),j} \cdot \ldots \cdot a_{\sigma(n),n} \qquad \square$$

Remarks:

·) Let $T \in \mathcal{L}(V)$ be an operator, then

$\det(z \cdot I_V - T)$ is the characteristic polynomial of T.

·) $\det(A) = \det(A^T)$

=> det is also an alternating, multilinear and normalized function of the rows of A.

=> Alternative way of computing $\det(A)$:

·) Bring A into upper-triangular form $B = \begin{pmatrix} b_{11} & & * \\ & \ddots & \\ 0 & & b_{nn} \end{pmatrix}$ using row operations

·) Let k be the number of times two rows are swapped.

·) $\det(A) = (-1)^k \det(B) = (-1)^k \, b_{11} \cdots b_{nn}$.