Lecture 39: Existence and uniqueness of the determinant

Last time: Determinant of matrices and operators

Determinant function $f: M_n(\mathbb{F}) \to \mathbb{F}$:

- alternating: $f(a_1|\dots|a_j|\dots|a_k|\dots|a_n) = 0$

  if $a_j = a_k$ for some $j<k$.

- multilinear: $f(\dots|a_i + \lambda b_i|\dots) = f(\dots|a_i|\dots) + \lambda f(\dots|b_i|\dots)$

- normalized: $f(e_1|\dots|e_n) = 1$

alternating + multilinear $\Rightarrow$ $f(\dots|a_i|\dots|a_j|\dots) = -f(\dots|a_j|\dots|a_i|\dots)$

$\boxed{\text{Prop}}$ For all $n \in \mathbb{N}$, there is exactly one determinant function

$\det: M_n(\mathbb{F}) \to \mathbb{F}$.

Proof: Existence of a determinant function

Induction on n:

$n=1$: $\det: M_1(\mathbb{F}) \to \mathbb{F}$, $\det(x) = x$

Let now $n \geq 2$ and $A \in M_n(\mathbb{F})$.

Denote by $\tilde{A}_{ij}$ the $(n-1)\times(n-1)$ matrix obtained from

removing the i-th row and j-th column of A.

Now define a function $D_n : M_n(\mathbb{F}) \to \mathbb{F}$ via

det fct on $M_{n-1}(\mathbb{F})$

$$D_n(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} D_{n-1}(\tilde{A}_{ij})$$

for fixed i $\qquad$ (i,j)-th elem. of A

We check:

·) D_n is multilinear: Let $A = (\alpha_1 | \dots | \alpha_k | \dots | \alpha_n)$

with $\alpha_k = \beta + e\gamma$, $\beta, \gamma \in \mathbb{F}^n$, $e \in \mathbb{F}$

Set $B = (\alpha_1 | \dots | \beta | \dots | \alpha_n)$
$C = (\alpha_1 | \dots | \gamma | \dots | \alpha_n)$

$\Bigg\}$ want to show: $D_n(A) = D_n(B) + e D_n(C)$

→ for $t \neq k$, we have $a_{st} = b_{st} = c_{st}$

→ $\forall i$ we have $\tilde{A}_{ik} = \tilde{B}_{ik} = \tilde{C}_{ik}$

→ for $j \neq k$, $D_{n-1}(\tilde{A}_{ij}) = D_{n-1}(\tilde{B}_{ij}) + e D_{n-1}(\tilde{C}_{ij})$

$$\Rightarrow D_n(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} D_{n-1}(\tilde{A}_{ij})$$

$$= \sum_{j \neq k} (-1)^{i+j} \overset{b_{ij}=c_{ij}}{\overset{\|}{a_{ij}}} \underbrace{D_{n-1}(\tilde{A}_{ij})}_{= D_{n-1}(\tilde{B}_{ij}) + e D_{n-1}(\tilde{C}_{ij})} + (-1)^{i+k} \overset{b_{ik}+e c_{ik}}{\overset{\|}{a_{ik}}} D_{n-1}(\underset{\tilde{B}_{ik} = \tilde{C}_{ik}}{\underset{\|}{\tilde{A}_{ik}}})$$

$$= \sum_{j=1}^{n} (-1)^{i+j} b_{ij} D_{n-1}(\tilde{B}_{ij}) + e \sum_{j=1}^{n} (-1)^{i+j} c_{ij} D_{n-1}(\tilde{C}_{ij})$$

$$= D_n(B) + e D_n(C) \checkmark$$

ii) alternating: let $A = (\alpha_1 | \dots | \alpha_k | \dots | \alpha_l | \dots | \alpha_n)$ with $\alpha_k = \alpha_l$ for some $k < l$.

Since D_{n-1} is alternating, $D_{n-1}(\tilde{A}_{ij}) = 0$ if $j \notin \{k, l\}$

$$D_n(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} D_{n-1}(\tilde{A}_{ij})$$

$$= (-1)^{i+k} a_{ik} D_{n-1}(\tilde{A}_{ik}) + (-1)^{i+l} \overset{a_{ik}}{\overset{\|}{a_{il}}} D_{n-1}(\tilde{A}_{il})$$

$$= (-1)^{i+k} a_{ik} \left(D_{n-1}(\tilde{A}_{ik}) + (-1)^{l-k} D_{n-1}(\tilde{A}_{il}) \right)$$

same columns in different order

obtain $\tilde{A}_{ik}$ from $\tilde{A}_{il}$ using $l-k-1$ column swaps

$$\Rightarrow D_{n-1}(\tilde{A}_{ik}) = (-1)^{l-k-1} D_{n-1}(\tilde{A}_{il})$$

$$\Rightarrow D_n(A) = (-1)^{i+k} a_{ik} \left((-1)^{l-k-1} D_{n-1}(\tilde{A}_{il}) + (-1)^{l-k} D_{n-1}(\tilde{A}_{il}) \right)$$

$$= 0 \checkmark$$

·) D_n is normalized:

$$D_n(I_n) = \sum_{j=1}^{n} (-1)^{i+j} \underbrace{(I_n)_{ij}}_{=\delta_{ij}} D_{n-1}\Big(\underbrace{(\widetilde{I_n})_{ij}}_{= I_{n-1} \text{ if } i=j}\Big)$$

$$= \underbrace{(-1)^{2i}}_{=1} D_{n-1}(I_{n-1}) = 1 \checkmark$$

<u>Uniqueness of the determinant</u>

Induction on n:

$n=1$: by linearity and normalization,

$x \in \mathbb{F}$: $\quad D_1(x) = D_1(x \cdot 1) = x\, D_1(1) = x \cdot 1 = x$

Let now $n \geq 2$, D_{n-1} the unique determinant function on $M_{n-1}(\mathbb{F})$, and D_n is some determinant function.

Know: if A is singular (non-invertible), then $D_n(A) = 0$ for any determinant function.

Hence, compute $D_n(A)$ for invertible A.

By taking linear combinations of columns and using multilinearity, we can assume w.l.o.g. that A has the form

$$\left[\begin{array}{c|c} A' & \begin{matrix} c_1 \\ \vdots \\ c_{n-1} \end{matrix} \\ \hline 0 \dots 0 & 1 \end{array}\right] \qquad (A' \in M_{n-1}(\mathbb{F}))$$

Define a function $f_{n-1} : M_{n-1}(\mathbb{F}) \to \mathbb{F}$, via

$$f_{n-1}(B) = D_n \left[\begin{array}{c|c} B & \begin{matrix} c_1 \\ \vdots \\ c_{n-1} \end{matrix} \\ \hline 0 \dots 0 & 1 \end{array}\right]$$

Because D_n is alternating and multilinear, f_{n-1} also has these properties.

By the proposition in the previous lecture, we have

$$f_{n-1}(B) = f_{n-1}(I_{n-1})\, D_{n-1}(B)$$

But $f_{n-1}(I_{n-1}) = D_n \left[\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & \begin{matrix} c_1 \\ \vdots \\ c_{n-1} \end{matrix} \\ \hline 0 \dots 0 & 1 \end{array}\right]$ $\qquad \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 \\ \vdots \\ c_{n-1} \\ 1 \end{pmatrix}$

$$= c_1 \underbrace{D_n \left[\begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} 1 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \dots 0 & 0 \end{array}\right]}_{=0} + D_n \left[\begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} 0 \\ c_2 \\ \vdots \\ c_{n-1} \end{matrix} \\ \hline 0 \dots 0 & 1 \end{array}\right]$$

$$\vdots$$

$$= D_n \left[\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \dots 0 & 1 \end{array}\right] = D_n(I_n) = 1$$

(by normalization)

$\Rightarrow D_n(A) = D_{n-1}(A')$, and hence D_n is uniquely determined by D_{n-1}. $\square$