Lecture 36: Jordan normal form

Last time: Characteristic and minimal polynomial

V complex finite-dim. VS

We first discuss special bases for nilpotent operators:

Ex.: Let $N: \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $\Rightarrow N^4 = 0$

$\Rightarrow N$ is nilpotent

take $v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, then $N(v) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $N^2(v) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $N^3(v) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow B = \{N^3(v), N^2(v), N(v), v\}$ is a basis of V, and

$$\mathcal{M}(N)_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Ex.: $M: (x_1, \dots, x_6)^T \mapsto (0, x_1, x_2, 0, x_4, 0)^T$

Observe that $M^3 = 0$, and the following is a basis for $\mathbb{R}^6$:

$v_1 = (1,0,0,0,0,0)^T$, $v_2 = (0,0,0,1,0,0)^T$, $v_3 = (0,0,0,0,0,1)^T$

$B = \{M^2(v_1), M(v_1), v_1, M(v_2), v_2, v_3\}$ is a basis for $\mathbb{R}^6$.

$$\mathcal{M}(M)_B = \left(\begin{array}{ccc|cc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We can prove that every nilpotent operator has such a matrix rep:

$\boxed{\text{Prop 8.55}}$ Let $N \in \mathcal{L}(V)$ be nilpotent. Then there exist vectors $v_1, \dots, v_n \in V$ and non-negative integers $m_1, \dots, m_n$ s.t.

i) $\{N^{m_1}(v), N^{m_1-1}(v), \dots, N(v_1), v_1, N^{m_2}(v_2), \dots, v_2, \dots, N(v_n), v_n\}$ is a basis for V.

ii) $N^{m_j+1}(v_j) = 0 \quad \forall j = 1, \dots, n$.

Proof: Induction on $\dim V$.

$\dim V = 1$: The only nilpotent operator is 0, and the claim follows by choosing any non-zero vector as the basis for V (and $m_1 = 0$)

assume now $\dim V > 1$:

Since N is nilpotent, N is not injective, hence not surjective, and $\dim \operatorname{im} N < \dim V$.

$\Rightarrow$ apply induction hypothesis to $N|_{\operatorname{im} N} \in \mathcal{L}(\operatorname{im} N)$:

There are vectors $v_1, \dots, v_n \in \operatorname{im} N$ and integers $m_1, \dots, m_n$, $m_i \geq 0$, s.t. $\{N^{m_1}(v_1), \dots, v_1, N^{m_2}(v_2), \dots, v_2, \dots, v_n\}$ is a basis for $\operatorname{im} N$, and $N^{m_j+1}(v_j) = 0 \quad \forall j$.

$v_j \in \operatorname{im} N \iff \exists\, u_j \in V$ s.t. $N(u_j) = v_j$.

Claim: $\{N^{m_1+1}(u_1), \dots, u_1, N^{m_2+1}(u_2), \dots, u_n\}^{(*)}$ is lin. indep.

Proof of claim: Let some linear combination of these vectors equal the zero vector. Then applying N on both sides and using $N^{k+1}(u_j) = N^k(v_j)$, we get a linear combination of the vectors $N^k(v_j)$ equal to zero. Since those are a basis for $\operatorname{im} N$, all coefficients of $N^k(v_j)$ are zero, except for the vectors

$$\underset{\substack{\| \\ N^{m_1}(v_1)}}{N^{m_1+1}(u_1)}, \dots, \underset{\substack{\| \\ N^{m_n}(v_n)}}{N^{m_n+1}(v_n)}.$$

$N^{m_j}(v_j)$ are also part of the basis for $\operatorname{im} N$, so their coefficients also have to vanish. $\Rightarrow$ $(*)$ linearly independent.

Now extend $(*)$ to a basis for V by adding vectors $w_1, \dots, w_p \in V$.

We have $N(w_j) \in \operatorname{im} N = \operatorname{span}\left(N^{m_1}(v_1), \dots, v_1, N^{m_2}(v_2), \dots, v_2, \dots, v_n\right)$

$$= N\left(\operatorname{span}\left(N^{m_1+1}(u_1), \dots, u_1, \dots, u_n\right)\right)$$

$\Rightarrow$ there is $x_j \in \operatorname{span}\left(N^{m_1+1}(u_1), \dots, u_1, \dots, u_n\right)$ s.t.

$$N(w_j) = N(x_j)$$

Set $u_{n+j} = w_j - x_j \rightarrow N(u_{n+j}) = N(w_j) - N(x_j) = 0$.

and furthermore $V = \operatorname{span}\big(N^{m_1+1}(u_1), \ldots, u_1, N^{m_2+1}(u_2), \ldots, u_2,$
$\ldots, u_n, u_{n+1}, \ldots, u_{n+p}\big)$

$\Rightarrow \{N^{m_1+1}(u_1), \ldots, u_1, \ldots, u_2, \ldots, u_n, u_{n+1}, \ldots, u_{n+p}\}$ is a basis for V with the desired properties. $\square$

In matrix form, Prop 8.55 says that there is a basis w.r.t. which N has a matrix representation with blocks of the form

$$\text{j-th block} \quad \begin{array}{c} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix} \\ \begin{array}{ccc} \uparrow \qquad\qquad & \uparrow & \nwarrow \\ N^{m_j}(v_j) \qquad\qquad & N(v_j) & v_j \end{array} \end{array}$$

$\boxed{\text{Prop 8.60}}$ Jordan normal form

Let V be a complex VS and $T \in \mathcal{L}_{\mathbb{C}}(V)$. Then there exists a basis of V s.t. T has matrix representation

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_p \end{pmatrix} \quad \text{where each } A_i \text{ is a block} \quad \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}$$

(λ_i are the eigenvalues of T)

Proof: Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T.

Consider $V = G(\lambda_1, T) \oplus \dots \oplus G(\lambda_m, T)$

We know $(T - \lambda_j I_V)|_{G(\lambda_j, T)}$ is nilpotent for each j.

By Prop 8.55 there is a basis for $G(\lambda_j, T)$ s.t. $(T - \lambda_j I_V)|_{G(\lambda_j,T)}$ has a matrix representation

$$\begin{pmatrix} N_1^{(j)} & & 0 \\ & \ddots & \\ 0 & & N_{k_j}^{(j)} \end{pmatrix} \text{ where each } N_i^{(j)} = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}.$$

$(G_j \equiv G(\lambda_j, T))$

$T|_{G_j} = (T - \lambda_j I_V)|_{G_j} + \lambda_j I_V|_{G_j}$ then has

a matrix rep. $\begin{pmatrix} A_1^{(j)} & & 0 \\ & \ddots & \\ 0 & & A_{k_j}^{(j)} \end{pmatrix}$ where $A_i^{(j)} = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}$.

merging all these bases for $G(\lambda_j, T)$ to a basis for V then gives the desired result. $\square$

Remarks: ·) JNF gives a special upper-triangular form of T

$\Rightarrow$ diagonal entries are eigenvalues

·) the geometric multiplicity $\dim \mathrm{Eig}(\lambda_j, T)$ of an eigenvalue λ_j is the number of "Jordan blocks" corresponding to λ_j.

·) the algebraic multiplicity $\dim G(\lambda_j, T)$ of an eigenvalue λ_j is the sum of all sizes of the Jordan blocks corresponding to λ_j.

Ex.: $T: (x_1, x_2, x_3, x_4)^T \to (2x_1 + x_3,\ 3x_2 - x_3 + x_4,\ -x_2 + 3x_3,\ 2x_4)$

$$\mathrm{JNF}(T) = \left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 4 \end{array}\right)$$

geometric mult. of $\lambda_1 = 2$ is 1

algebraic mult. of $\lambda_1 = 2$ is 3

·) T is diag'ble $\Leftrightarrow$ alg. mult. = geom. mult. for all eigenvalues

$\Rightarrow$ all Jordan blocks have size 1×1, and the JNF is the diagonal form of T.