Lecture 34: Generalized eigenspaces and multiplicities of eigenvalues

Last time: Generalized eigenvectors and nilpotent operators

Goal for today: For every operator $T \in \mathcal{L}_{\mathbb{C}}(V)$, V has a basis consisting of generalized eigenvectors of T.

Prop Let $T \in \mathcal{L}_{\mathbb{F}}(V)$ be an operator on an arbitrary VS V.

i) $V = \ker T^n \oplus \operatorname{im} T^n$ where $n = \dim V$.

ii) For every polynomial $p \in P(\mathbb{F})$, the subspaces $\ker p(T)$ and $\operatorname{im} p(T)$ are invariant under T.

Proof: i) Let $v \in \ker T^n \cap \operatorname{im} T^n$:

Then $T^n(v) = 0$, and there exists $u \in V$ s.t. $v = T^n(u)$.

$\Rightarrow T^n(v) = 0 = T^n(T^n(u)) = T^{2n}(u)$

$\Rightarrow u \in \ker T^{2n} = \ker T^n$ (by Prop 8.4),

which means $T^n(u) = v = 0 \Rightarrow \ker T^n \cap \operatorname{im} T^n = \{0\}$.

Hence, $\ker T^n + \operatorname{im} T^n = \ker T^n \oplus \operatorname{im} T^n$

By the dimension formula for linear maps, $\dim V = \dim \ker T^n + \dim \operatorname{im} T^n$

$\Rightarrow V = \ker T^n \oplus \operatorname{im} T^n$.

ii) for a polynomial $p = a_0 + a_1 x + \ldots + a_d x^d$, we have

$$p(T) = a_0 I_V + a_1 T + \ldots + a_d T^d \in \mathcal{L}(V)$$

Let $v \in \ker p(T)$, then $p(T)(v) = 0$, and

$$p(T)(T(v)) = T(p(T)(v)) = T(0) = 0 \Rightarrow T(v) \in \ker p(T).$$

Let $v \in \operatorname{im} p(T)$, then there is $u \in V$ with $p(T)(u) = v$.

Then $\quad T(v) = T(p(T)(u)) = p(T)(T(u)) \in \operatorname{im} p(T).$ $\square$

$\boxed{\text{Prop 8.21}}$ Let V be a VS over $\mathbb{C}$, $T \in \mathcal{L}_{\mathbb{C}}(V)$ an operator with distinct eigenvalues $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$. Then:

i) $V = G(\lambda_1, T) \oplus \ldots \oplus G(\lambda_m, T)$

ii) Each $G(\lambda_i, T)$ is invariant under T.

iii) Each $(T - \lambda_i I_V)\big|_{G(\lambda_i, T)}$ is nilpotent.

iv) V has a basis consisting of generalized eigenvectors of T.

<u>Proof</u>: ii) follows from the above Prop. (ii) with $p(z) = (z - \lambda_j)^n$,

since $G(\lambda_j, T) = \ker (T - \lambda_j I_V)^n$, $n = \dim V$ (Prop 8.11).

iii) easy consequence of $G(\lambda_j, T) = \ker (T - \lambda_j I)^n$

iv) follows easily from i) (choose bases for each $G(\lambda_j, T)$)

i) induction on $n = \dim V$.

$n = 1$: there is nothing to prove.

Let now $n > 1$. Since $T \in \mathcal{L}_{\mathbb{C}}(V)$ is an operator over a complex VS,

T has eigenvalues, and $m \geq 1$ (# distinct eigenvalues)

Setting $G(\lambda_1, T) = \ker (T - \lambda_1 I_V)^n$ and $U = \operatorname{im} (T - \lambda_1 I_V)^n$,

we have $V = G(\lambda_1, T) \oplus U$ by previous Prop (i)

and U is invariant under T by Prop (ii).

Since $\dim G(\lambda_1, T) \geq 1$, we have $\dim U < n = \dim V$.

$\Rightarrow$ use induction hypothesis for U and $T|_U$:

$$U = G(\lambda_2, T|_U) \oplus \ldots \oplus G(\lambda_m, T|_U),$$

where $\lambda_2, \ldots, \lambda_m$ are the rest of the distinct eigenvalues

$\Rightarrow V = G(\lambda_1, T) \oplus G(\lambda_2, T|_U) \oplus \ldots \oplus G(\lambda_m, T|_U)$

to show: $G(\lambda_j, T|_U) = G(\lambda_j, T)$

($\subseteq$) clear that $G(\lambda_j, T|_U) \subseteq G(\lambda_j, T)$ ✓

($\supseteq$) Let $v \in G(\lambda_j, T) \subseteq V$, then there are $v_1 \in G(\lambda_1, T)$ and $u \in U$,

s.t. $v = v_1 + u$.

$\Rightarrow$ there are $v_2, \ldots, v_m$, $v_i \in G(\lambda_i, T|_U) \subseteq G(\lambda_i, T)$ with

$$v = v_1 + v_2 + \ldots + v_m$$

$\Rightarrow$ there are $v_2, \dots, v_m$, $v_i \in G(\lambda_i, T|_U) \subseteq G(\lambda_i, T)$ with
$G(\lambda_j, T) \ni v = v_1 + v_2 + \dots + v_m$

$$0 = v_1 + \dots + v_{j-1} + (v_j - v) + \dots + v_m$$

The v_i are either 0 or linearly independent (because they are from generalized eigenspaces corresp. to <u>distinct</u> eigenvalues), and so in particular, $v_1 = 0$.

$$v = v_1 + u = u \in U = G(\lambda_2, T|_U) \oplus \dots \oplus G(\lambda_m, T|_U)$$

Since $v \in G(\lambda_j, T)$, we have $v \in G(\lambda_j, T|_U)$

$\Rightarrow G(\lambda_j, T) \subseteq G(\lambda_j, T|_U)$

$\Rightarrow G(\lambda_j, T) = G(\lambda_j, T|_U)$. $\square$

Multiplicities of eigenvalues

$\boxed{\text{Def. 8.24}}$ Let $T \in \mathcal{L}(V)$, then the multiplicity of an eigenvalue λ of T is defined as $\dim G(\lambda, T) = \dim \ker (T - \lambda I_V)^n$, where $n = \dim V$.

$\boxed{\text{Prop 8.26}}$ Let V be a complex vector space and $T \in \mathcal{L}_{\mathbb{C}}(V)$. Then the sum of the multiplicities of the eigenvalues of T is equal to the dimension of V.

Proof: Follows immediately from Prop 8.21 i).

Remark: The following terminology is very common:

algebraic multiplicity of an eigenvalue λ: $\dim G(\lambda, T)$

$$= \dim \ker (T - \lambda I_V)^n$$

geometric multiplicity of an eigenvalue λ: $\dim Eig(\lambda, T)$

$$= \dim \ker (T - \lambda I_V)$$

multiplicity from Def 8.29 $\leftrightarrow$ algebraic multiplicity.

$\boxed{\text{Prop 8.29}}$ Let $T \in \mathcal{L}_{\mathbb{C}}(V)$ with distinct eigenvalues $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ of multiplicities $d_1, \dots, d_m$ (i.e., $d_1 + \dots + d_m = \dim V$)

Then V has a basis $\mathcal{B}$ w.r.t. which T has the following matrix representation:

$$M(T) = \left(\begin{array}{c|c|c|c} A_1 & 0 & 0 & \\ \hline 0 & A_2 & 0 & \\ \hline 0 & & \ddots & \\ \hline & & & A_m \end{array} \right)$$

where each A_j is a $d_j \times d_j$-matrix of the form

$$A_j = \begin{pmatrix} \lambda_j & & * \\ & \ddots & \\ 0 & & \lambda_j \end{pmatrix} \quad \text{(upper-triangular)}$$

Proof: Let $V = G(\lambda_1, T) \oplus \dots \oplus G(\lambda_m, T)$

Each $(T - \lambda_j I_V)|_{G(\lambda_j, T)}$ is nilpotent $\left(\left((T-\lambda_j I_V)^n\right)|_{G(\lambda_j,T)} = 0\right.$

$\left.\text{for } n = \dim V \right)$,

so by Prop 8.19 there exists a basis B_j for $G(\lambda_j, T)$ s.t.

$$\mathcal{M}\left((T - \lambda_j I_V)|_{G(\lambda_j, T)} \right) = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

Then w.r.t. the same basis B_j, we have $\quad (G = G(\lambda_j, T))$

$$\mathcal{M}(T|_G) = \mathcal{M}((T - \lambda_j I_V)|_G) + \mathcal{M}(\lambda_j I_V|_G)$$

$$= \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} + \begin{pmatrix} \lambda_j & & 0 \\ & \ddots & \\ 0 & & \lambda_j \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_j & & * \\ & \ddots & \\ 0 & & \lambda_j \end{pmatrix} = A_j$$

Merging all B_j's into one basis for V gives the desired result. $\square$