Lecture 32: Polar decomposition and singular value decomposition

Last time: Positive operators and isometries

$T = T^*$ with $\langle T(v), v\rangle \geq 0 \ \forall v \in V$

$\|S(v)\| = \|v\| \ \forall v \in V$

Recall: isometric operators on V have eigenvalues with modulus 1
($\lambda \in \mathbb{C}$ with $|\lambda| = 1$)

For every complex number $z \in \mathbb{C}$, $z \neq 0$, we can write

$$z = \frac{z}{|z|} \cdot |z| = \frac{z}{|z|} \sqrt{\bar{z} z} \qquad (*)$$

where $\left|\frac{z}{|z|}\right| = 1$ and $\sqrt{\bar{z} z} \geq 0$.

Polar decomposition: operator version of $(*)$ (with $\bar{z} \leftrightarrow T^*$)

Notation: If $P \in \mathcal{L}(V)$ is positive, we denote by $\sqrt{P}$ its unique positive square root (if P has eigenvalues $\lambda_i \geq 0$, then $\sqrt{P}$ has eigenvalues $+\sqrt{\lambda_i}$)

$\boxed{\text{Prop 7.45}}$ Polar decomposition

Let $T \in \mathcal{L}(V)$ be arbitrary. Then there exists an isometry $S \in \mathcal{L}(V)$ (if $\mathbb{F} = \mathbb{C}$, then S is unitary, if $\mathbb{F} = \mathbb{R}$ then S is orthogonal) s.t.

$$T = S\sqrt{T^*T}.$$

Proof: T^*T is positive for every $T \in \mathcal{L}(V)$, so $\sqrt{T^*T}$ is well-defined.

Let $v \in V$ be arbitrary, then

$$\|T(v)\|^2 = \langle T(v), T(v)\rangle = \langle T^*T(v), v\rangle = \langle \sqrt{T^*T}\sqrt{T^*T}(v), v\rangle$$

$$= \langle \sqrt{T^*T}(v), \sqrt{T^*T}(v)\rangle = \|\sqrt{T^*T}(v)\|^2 \qquad (**)$$

$\sqrt{T^*T}$ is self-adjoint

Let us define a map $S_1 : \operatorname{im}\sqrt{T^*T} \to \operatorname{im} T$

$$S_1\left(\sqrt{T^*T}(v)\right) = T(v)$$

$v \in V$

The idea of the proof is to show that S_1 is a well-defined linear map that can be extended to an isometry S satisfying

$$S\sqrt{T^*T} = T \quad \text{(i.e., the claim of the proposition)}$$

·) Well-definedness: Let $v_1, v_2 \in V$ be s.t. $\sqrt{T^*T}(v_1) = \sqrt{T^*T}(v_2)$

to show: $T(v_1) = T(v_2)$

we have $\|T(v_1) - T(v_2)\| = \|T(v_1 - v_2)\|$

$$= \|\sqrt{T^*T}(v_1 - v_2)\|$$

$$= \|\sqrt{T^*T}(v_1) - \sqrt{T^*T}(v_2)\|$$

$$= 0 \quad \Rightarrow T(v_1) = T(v_2) \checkmark$$

·) Linearity : easy to check (follows from linearity of $\sqrt{T^*T}$ and T)

Since $\|\sqrt{T^*T}(v)\| = \|T(v)\| \; \forall v \in V$, we have $\ker\sqrt{T^*T} = \ker T$

$\Rightarrow \dim \operatorname{im}\sqrt{T^*T} = \dim \operatorname{im} T$, $\dim(\operatorname{im}\sqrt{T^*T})^\perp = \dim(\operatorname{im}T)^\perp$

(recall: $S_1 : \operatorname{im}\sqrt{T^*T} \to \operatorname{im}T$)

Let $\{w_1, \dots, w_m\}$ and $\{\tilde{w}_1, \dots, \tilde{w}_m\}$ be ONBs for $(\operatorname{im}\sqrt{T^*T})^\perp$ and $(\operatorname{im}T)^\perp$, resp., and define a linear map

$$S_2 : (\operatorname{im}\sqrt{T^*T})^\perp \to (\operatorname{im}T)^\perp$$

$$S_2(w_i) = \tilde{w}_i \quad \forall i = 1, \dots, m$$

(S_2 is an isometry, since it maps an ONB to an ONB)

Since $V = \operatorname{im}\sqrt{T^*T} \oplus (\operatorname{im}\sqrt{T^*T})^\perp$, we can write any vector $v \in V$ in a <u>unique</u> way as $v = u + w$, with $u \in \operatorname{im}\sqrt{T^*T}$, $w \in (\operatorname{im}\sqrt{T^*T})^\perp$

Define a linear map $S : V \to V$ by

$$S(v) = S_1(u) + S_2(w)$$

Then, $S\sqrt{T^*T}(v) = S_1(\sqrt{T^*T}(v)) = T(v) \quad \forall v \in V$,

$$\text{so } S\sqrt{T^*T} = T$$

Remains to be shown: S is an isometry.

$\forall v \in V$: $\quad \|S(v)\| = \|S_1(u) + S_2(w)\| = \|S_1(u)\| + \|S_2(w)\|$

$$\overset{!}{=} \|u\| + \|w\| = \|v\|$$

where $\underbrace{u \in \operatorname{im}\sqrt{T^*T}}$, $w \in (\operatorname{im}\sqrt{T^*T})^\perp$

$\exists \tilde{u} \in V$ with $u = \sqrt{T^*T}(\tilde{u})$:

$$\|S_1(u)\| = \|S_1(\sqrt{T^*T}(\tilde{u}))\| = \|T(\tilde{u})\| \overset{(**)}{=} \|\sqrt{T^*T}(\tilde{u})\| = \|u\|$$

$$\|S_2(\underbrace{w}_{\sum_{i=1}^m a_i w_i})\|^2 = \|S_2(\sum_{i=1}^m a_i w_i)\|^2 = \|\sum_{i=1}^m a_i \tilde{w}_i\|^2 = \sum_{i=1}^m |a_i|^2 = \|w\|$$

$$\Rightarrow \|S(v)\| = \|S_1(u)\| + \|S_2(w)\|$$

$$= \|u\| + \|w\| = \|v\| \quad \forall v \in V$$

$\Rightarrow S$ is an isometry $\square$

Singular value decomposition

$\boxed{\text{Def 7.49}}$ Let $T \in \mathcal{L}(V)$. The singular values of T are the eigenvalues of the positive operator $\sqrt{T^*T}$.

$\boxed{\text{Prop}}$ Singular value decomposition for operators

Let $T \in \mathcal{L}(V)$ be an op. with singular values $s_1, \dots, s_n \geq 0$ $(n = \dim V)$.

Then there exist ONBs $\mathcal{B}$ and $\tilde{\mathcal{B}}$, s.t.

$$\mathcal{M}(T)_{\mathcal{B},\tilde{\mathcal{B}}} = \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{pmatrix}$$

Proof: Let $s_1, \dots, s_n$ be the eigenvalues of $\sqrt{T^*T}$

Since $\sqrt{T^*T}$ is positive:

- $s_i \geq 0 \quad \forall i = 1, \dots, n$
- there is an ONB $B = \{v_1, \dots, v_n\}$ of V consisting of eigenvectors of $\sqrt{T^*T}$: $\quad \sqrt{T^*T}(v_i) = s_i v_i \quad \forall i = 1, \dots, n. \quad (*)$

  (by the complex/real spectral theorem)

Let $T = S\sqrt{T^*T}$ be the polar decomposition of T, where

$S \in \mathcal{L}(V)$ is an isometry.

Apply S to $(*)$: $\quad T(v_i) = S\sqrt{T^*T}(v_i) = S(s_i v_i) = s_i S(v_i)$

Define $w_i = S(v_i)$, then $\tilde{B} = \{w_1, \dots, w_n\}$ is also an ONB (because S is an isometry), and $T(v_i) = s_i w_i \Rightarrow \mathcal{M}(T)_{B, \tilde{B}} = \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{pmatrix}$.

$\square$

Remarks:

- singular values s_i are the eigenvalues of $\sqrt{T^*T}$, and hence equal to the positive square roots of the eigenvalues of T^*T.
- SVD is very useful in applications!
- SVD can be generalized to arbitrary linear maps $T \in \mathcal{L}(V, W)$ between inner product spaces V, W.