Lecture 31: Positive operators and isometries

Last time: Spectral theorems

$\boxed{\text{Def 7.31}}$ An operator $T \in \mathcal{L}(V)$ on an inner product space V is called positive (or positive semi-definite), if T is self-adjoint, and $\langle T(v), v \rangle \geq 0 \quad \forall v \in V$.

Note: If $\mathbb{F} = \mathbb{C}$, then $\langle T(v), v \rangle \geq 0 \ \forall v \in V$ implies self-adjointness (thm), but not over $\mathbb{F} = \mathbb{R}$.

Ex.: Orthogonal projections onto some subspace $U \leq V$.

$\forall v \in V$: $v = u + w$, $u \in U$, $w \in U^\perp$, and $P_U(v) = u$.

$$\langle P_U(v), v \rangle = \langle P_U(u+w), u+w \rangle = \langle u, u+w \rangle = \underbrace{\langle u, u \rangle}_{\geq 0} + \underbrace{\langle u, w \rangle}_{=0}$$

$$\geq 0$$

$\boxed{\text{Def}}$ An operator $R \in \mathcal{L}(V)$ is called a square root of $T \in \mathcal{L}(V)$ if $R^2 = T$.

Ex.: Let P be a projection: $P^2 = P$, then P is its own square root.

Prop 7.35 For $T \in \mathcal{L}(V)$, TFAE:

i) T is positive

ii) T is self-adjoint, and all eigenvalues of T are non-negative.

iii) T has a positive square root.

iv) T has a self-adjoint square root.

v) there exists an operator $R \in \mathcal{L}(V)$ s.t. $T = R^* R$.

__Proof:__ i) => ii) T is self-adjoint by def.

Let $v \neq 0$ be an eigenvector of T with eigenvalue λ:

$$0 \leq \langle T(v), v \rangle = \langle \lambda v, v \rangle = \lambda \underbrace{\langle v, v \rangle}_{>0} \implies \lambda \geq 0.$$

ii) => iii) By the spectral theorems (for self-adjoint op's over $\mathbb{C}$ or $\mathbb{R}$), V has an ONB consisting of eigenvectors of T.

Let $\{v_1, \dots, v_n\}$ be the ONB of eigenvec's with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$, $\lambda_i \geq 0$ $\forall i$.

Define a map $R: v_i \mapsto \sqrt{\lambda_i}\, v_i$, then

- R is positive (HW)
- $R^2(v_i) = R(R(v_i)) = R(\sqrt{\lambda_i}\, v_i) = \sqrt{\lambda_i}\, R(v_i) = \sqrt{\lambda_i}\sqrt{\lambda_i}\, v_i = \lambda_i v_i$
  $= T(v_i) \implies R^2 = T$ because $\{v_1, \dots, v_n\}$ is a basis.

iii) $\Rightarrow$ iv) clear, since R positive $\Rightarrow$ R is self-adjoint.

iv) $\Rightarrow$ v) if $T = R^2$ with $R = R^*$, then $T = R^* R = R R^*$

v) $\Rightarrow$ i) if $T = R^* R$ for some $R \in \mathcal{L}(V)$, then

·) $T^* = (R^* R)^* = R^* (R^*)^* = R^* R = T$

·) $\forall v \in V$, $\langle T(v), v \rangle = \langle R^* R(v), v \rangle = \langle R(v), R(v) \rangle \geq 0$.

$\square$

Note: A positive operator can have many square roots, but there is a unique positive square root (given by the map constructed above in the proof). See Prop 7.36 in the textbook (Axler).

Isometries

Isometries are maps that preserve norms / inner products:

$\boxed{\text{Def}}$ Let V, W be normed vector spaces, i.e., vector spaces over $\mathbb{C}$ or $\mathbb{R}$ with norms $\|.\|_V$ and $\|.\|_W$. Then a linear map $S \in \mathcal{L}(V, W)$ is called an isometry, if $\|S(v)\|_W = \|v\|_V$ $\forall v \in V$.

Prop Let V, W be inner product spaces with norms $\|.\|_V = \sqrt{<.,.>_V}$ and $\|.\|_W = \sqrt{<.,.>_W}$. Then $S \in \mathcal{L}(V,W)$ is an isometry if and only if $< S(u), S(v) > = < u, v > \ \forall u, v \in V$.

Proof: ($\Leftarrow$) $\|S(v)\|_W^2 = < S(v), S(v) > = < v, v > = \|v\|_V^2 \ \forall v \in V \Rightarrow$ claim.

($\Rightarrow$) Use the polarization identities:

if $\mathbb{F} = \mathbb{R}$, then $< u, v > = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right)$

if $\mathbb{F} = \mathbb{C}$, then $< u, v > = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2\right)$

In either case, linearity of S and $\|S(x)\| = \|x\| \ \forall x \in V$ proves that also $< u, v > = < S(u), S(v) > \ \forall u, v \in V$. $\square$

Prop Characterization of isometries in inner product spaces

Let V, W be inner product spaces with associated norms $\|.\|_V$ and $\|.\|_W$, and let $S \in \mathcal{L}(V, W)$ be an isometry. Then:

i) $S^* S = I_V$

ii) S is injective.

iii) $\{S(v_1), \dots, S(v_n)\}$ are ON if $\{v_1, \dots, v_n\}$ are orthonormal.

Proof: i) S isometry, then $\forall u, v \in V$,

$$\langle u, v \rangle = \langle S(u), S(v) \rangle = \langle u, S^*S(v) \rangle$$

$$\Rightarrow \langle u, v - S^*S(v) \rangle = 0 \quad \forall u, v \in V \quad (*)$$

Choose $u = v - S^*S(v)$, then $(*)$ implies that $v = S^*S(v)$ $\forall v \in V$

$$\Rightarrow S^*S = I_V.$$

ii) Let $v \in \ker S$, i.e., $S(v) = 0$, then

$$0 = \|0\| = \|S(v)\| = \|v\| \Rightarrow v = 0 \Rightarrow \ker S = \{0\}.$$

iii) Let $\{v_1, \dots, v_n\}$ be a set of ON vectors in V $(\langle v_i, v_j \rangle = \delta_{ij})$

Set $w_i := S(v_i)$, then $\langle w_i, w_j \rangle = \langle S(v_i), S(v_j) \rangle$

$$= \langle v_i, v_j \rangle = \delta_{ij}$$

$\Rightarrow \{S(v_1), \dots, S(v_n)\}$ are ON as well. $\square$

Unitary and orthogonal operators

Set now $V = W$ and let $S: V \to V$ be an isometry.

For $\mathbb{F} = \mathbb{C}$, S is called a unitary operator.

In addition to the properties above, we have:

.) S is invertible, and $S^{-1} = S^*$: Every injective operator is invertible, and $S^*S = I_V$ shows that $S^{-1} = S^*$.

- $S^*S = SS^* = I_V$, and $S^* : V \to V$ is also an isometry.
- Unitary operators map an orthonormal basis to an orthonormal basis.
- S is diagonalizable, and all its eigenvalues have absolute value 1:

  Since $S^*S = SS^* = I_V$, the operator S is normal and hence diag'ble by the complex spectral theorem (Prop 7.24).

  Let λ be an eigenvalue of S with eigenvec. v:

$$\underbrace{\langle v, v\rangle}_{\neq 0} = \langle S(v), S(v)\rangle = \langle \lambda v, \lambda v\rangle = \lambda\bar{\lambda}\langle v, v\rangle = |\lambda|^2\langle v, v\rangle$$

$$\Rightarrow |\lambda|^2 = 1 \Rightarrow |\lambda| = 1.$$

For $\mathbb{F} = \mathbb{R}$, an isometric operator S is called an orthogonal operator.

- S is invertible, and $S^{-1} = S^*$, $SS^* = S^*S = I_V$.
- Let $\mathcal{B}$ be an orthonormal basis, then $M = \mathcal{M}(S)_{\mathcal{B},\mathcal{B}}$ satisfies

  $MM^T = M^TM = I_n$ ($n = \dim V$, $M^* = \bar{M}^T = M^T$ for real matrices)

- S is again diagonalizable, but only over $\mathbb{C}$.

  Over $\mathbb{R}$, eigenvalues may not exist for orthogonal operators:

  E.g., $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for $\theta \neq k\pi$, $k \in \mathbb{Z}$

  rotation.