Lecture 30: Spectral theorems

Last time: Self-adjoint and normal operators

- $T \in \mathcal{L}(V)$ is self-adjoint if $T = T^*$.
- $T \in \mathcal{L}(V)$ is normal, if $TT^* = T^*T$

Every self-adjoint operator is normal, since then $TT^* = T^2 = T^*T$.

Prop 7.21 Let $T \in \mathcal{L}(V)$ be a normal operator and $v \neq 0$ an eigenvector of T with eigenvalue λ. Then v is also an eigenvector of T^* with eigenvalue $\bar{\lambda}$.

Proof: If T is normal, then also $T - \lambda I_V$ is normal:

$$(T - \lambda I)^* (T - \lambda I_V) = (T - \lambda I_V)(T - \lambda I_V)^*$$

Using Prop 7.20 from last time:

$$0 = \|(T - \lambda I_V)(v)\| = \|(T - \lambda I_V)^*(v)\| = \|(T^* - \bar{\lambda} I_V)(v)\|$$

and hence $(T^* - \bar{\lambda} I_V)(v) = 0$, i.e., v is also an eigenvec of T^* with eigenval $\bar{\lambda}$. $\square$

$\boxed{\text{Prop 7.22}}$ Let $T \in \mathcal{L}(V)$ be normal. Then eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof: Let $\alpha, \beta,$ $^{\alpha \neq \beta,}$ be eigenvalues with corr. eigenvectors u, v.

Then
$$(\alpha - \beta) \langle u, v \rangle = \alpha \langle u, v \rangle - \beta \langle u, v \rangle$$
$$= \langle \alpha u, v \rangle - \langle u, \bar{\beta} v \rangle$$
$$= \langle T(u), v \rangle - \langle u, T^*(v) \rangle = 0$$

By assumption, $\alpha \neq \beta$, and hence $\langle u, v \rangle = 0$. $\square$

Remark: Prop 7.22 applies in particular to self-adjoint operators.

$\boxed{\text{Prop 7.24}}$ Complex spectral theorem

Let $T \in \mathcal{L}_{\mathbb{C}}(V)$ be an operator over a complex inner product space V.

TFAE: i) T is normal

ii) V has an ONB consisting of eigenvectors of T.

iii) T is diagonalizable w.r.t. an ONB.

Proof: first: i) $\Leftrightarrow$ iii)

iii) $\Rightarrow$ i) Let B be an ONB s.t. $\mathcal{M}(T)_{B,B}$ is diagonal.

Then $\mathcal{M}(T^*)_{B,B} = \mathcal{M}(T)^*_{B,B}$ is clearly also diagonal, and

$\mathcal{M}(T)_{B,B}\, \mathcal{M}(T^*)_{B,B} = \mathcal{M}(T^*)_{B,B}\, \mathcal{M}(T)_{B,B} \Rightarrow TT^* = T^*T \Rightarrow$ i)

i) $\Rightarrow$ iii) By Schur's theorem (Prop 6.38), every $T \in \mathcal{L}_{\mathbb{C}}(V)$

has an upper-triangular matrix rep. w.r.t. an ONB $B = \{f_1, \dots, f_n\}$

$\langle f_i, f_j \rangle = \delta_{ij}$

$$M(T)_{B,B} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 \cdots 0 & & a_{nn} \end{pmatrix}$$

We have $T(f_1) = a_{11} f_1 \Rightarrow \|T(f_1)\|^2 = |a_{11}|^2$

Since $M(T^*)_{B,B} = M(T)^*_{B,B} = \begin{pmatrix} \bar{a}_{11} & & 0 \\ \vdots & \bar{a}_{12} & \\ \vdots & & \ddots \\ \bar{a}_{1n} & \cdots & \bar{a}_{nn} \end{pmatrix}$,

$(|z| = |\bar{z}|)$

$T^*(f_1) = \bar{a}_{11} f_1 + \bar{a}_{12} f_2 + \dots + \bar{a}_{1n} f_n$ and $\|T^*(f_1)\| = |a_{11}|^2 + \dots + |a_{1n}|^2$

But T is normal, so by Prop 7.20, $\|T(f_1)\|^2 = \|T^*(f_1)\|^2$,

and hence $|a_{12}|^2 = \dots = |a_{1n}|^2 = 0 \Rightarrow a_{12} = \dots = a_{1n} = 0$

For the second row of $M(T)$: $T(f_2) = \underset{\overset{\|}{0}}{a_{12}} f_1 + a_{22} f_2 = a_{22} f_2$

$\Rightarrow \|T(f_2)\|^2 = |a_{22}|^2$

$T^*(f_2) = \bar{a}_{22} f_2 + \bar{a}_{23} f_3 + \dots + \bar{a}_{2n} f_n \Rightarrow \|T^*(f_2)\|^2 = |a_{22}|^2 + \dots + |a_{2n}|^2$

Since $\|T(f_2)\|^2 = \|T^*(f_2)\|^2$, $|a_{23}|^2 = \dots = |a_{2n}|^2 = 0$

$\Rightarrow a_{23} = \dots = a_{2n} = 0$.

continue in this fashion for all rows $\Rightarrow$ all off-diagonal entries of $M(T)$ are 0

$\Rightarrow M(T)$ is diagonal w.r.t. an ONB $\mathcal{B}$.

ii) $\Leftrightarrow$ iii) easy to check. $\square$

The complex spectral theorem crucially depends on Schur's thm, which depends on the existence of an upper-triangular matrix rep, which in turn depends on the existence of eigenvalues

$\Rightarrow$ not guaranteed over real inner product spaces.

However, self-adjoint operators over real inner product spaces do always have eigenvalues: see Prop. 7.27 in the textbook (Axler)

Before we prove the real spectral theorem:

$\boxed{\text{Prop 7.28}}$ Let $T \in \mathcal{L}(V)$ be a self-adjoint operator, and $U \leq V$ a subspace invariant under T. Then:

i) $U^\perp$ is invariant under T.

ii) $T|_U \in \mathcal{L}(U)$ is self-adjoint.

iii) $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ is self-adjoint.

Proof: i) Let $v \in U^\perp$ and $u \in U$:

$$\langle T(v), u\rangle \underset{T=T^*}{=} \langle v, T(u)\rangle \underset{T(u)\in U}{=} 0 \quad \Rightarrow T(v) \in U^\perp$$

$\Rightarrow U^\perp$ is T-invariant.

ii) $u, v \in U$: $\langle T|_U(u), v\rangle = \langle T(u), v\rangle = \langle u, T(v)\rangle$

$$= \langle u, T|_U(v)\rangle$$

$\Rightarrow T|_U$ is self-adjoint.

iii) replace U in (ii) by $U^\perp$ (valid by i)) $\square$

$\boxed{\text{Prop 7.29}}$ Real spectral theorem

Let $T \in \mathcal{L}_{\mathbb{R}}(V)$ be an operator on a <u>real</u> inner product space.

TFAE: i) T is self-adjoint

ii) V has an ONB consisting of eigenvectors of T.

iii) T is diagonalizable w.r.t. an ONB.

Proof: i) $\Rightarrow$ ii) Induction on $\dim V$.

$\dim V = 1$: nothing to prove.

Induction step: Since $T \in \mathcal{L}_{\mathbb{R}}(V)$ is self-adjoint, there is an eigenvector $v \in V$ by Prop 7.27 in the textbook. Set $\tilde{v} = \frac{1}{\|v\|} v$, and $U = \langle v \rangle$. Since $\dim U = 1$, $\dim U^{\perp} = \dim V - 1$ and $U^{\perp}$ is invariant under T by Prop 7.28, $T|_{U^{\perp}}$ is again self-adjoint.

By the induction hypothesis, $U^{\perp}$ has an ONB consisting of eigenvectors of T. Add $\tilde{v}$ to this basis $\Rightarrow$ ONB for V consisting of eigenvectors of T $\Rightarrow$ ii)

ii) $\Rightarrow$ iii) $\Rightarrow$ i) is an easy exercise. $\square$