Lecture 28: Adjoint maps

Last time: Orthogonal projections

Standing assumption: V, W are finite-dim. inner product spaces over $\mathbb{C}$.

$\boxed{\text{Def 7.2}}$ Adjoint map

Let $T \in \mathcal{L}_{\mathbb{C}}(V, W)$. The adjoint of T, denoted by T^*,

is a map $W \to V$ defined via

$$(*) \qquad \langle T(v), w \rangle = \langle v, T^*(w) \rangle \quad \text{for all } v \in V, w \in W.$$

$\boxed{\text{Prop}}$ T^* is a well-defined, linear map $W \to V$ $\left(T^* \in \mathcal{L}_{\mathbb{C}}(W, V) \right)$.

Furthermore, it is uniquely defined by $(*)$.

Proof: Well-defined: Let $w \in W$ be arbitrary and assume that

$v_1 = T^*(w)$ and $v_2 = T^*(w)$ for some $v_1, v_2 \in V$.

To show: $v_1 = v_2$.

$\forall v \in V$, $\langle v, v_1 \rangle = \langle v, T^*(w) \rangle$ and $\langle v, v_2 \rangle = \langle v, T^*(w) \rangle$

$\Rightarrow 0 = \langle v, v_1 \rangle - \langle v, v_2 \rangle = \langle v, v_1 - v_2 \rangle \quad \forall v \in V$

Choose $v = v_1 - v_2 \in V$, then $0 = \langle v_1 - v_2, v_1 - v_2 \rangle$

$\Rightarrow v_1 - v_2 = 0 \Rightarrow v_1 = v_2$ ✓

Linearity: Let $w_1, w_2 \in W$, $\lambda_1, \lambda_2 \in \mathbb{C}$

To show: $T^*(\lambda_1 w_1 + \lambda_2 w_2) = \lambda_1 T^*(w_1) + \lambda_2 T^*(w_2)$

Let $v \in V$ be arbitrary: $< v, T^*(\lambda_1 w_1 + \lambda_2 w_2) >$

$$= < T(v), \lambda_1 w_1 + \lambda_2 w_2 >$$

$$= \bar{\lambda}_1 < T(v), w_1 > + \bar{\lambda}_2 < T(v), w_2 >$$

$$= \bar{\lambda}_1 < v, T^*(w_1) > + \bar{\lambda}_2 < v, T^*(w_2) >$$

$$= < v, \lambda_1 T^*(w_1) + \lambda_2 T^*(w_2) > \implies \checkmark$$

Uniqueness: Let S be another map s.t. $< v, S(w) > = < T(v), w >$ for all $v \in V$, $w \in W$.

$\implies < v, S(w) > = < v, T^*(w) >$

or $0 = < v, S(w) - T^*(w) > \quad \forall v \in V$

Choosing $v = S(w) - T^*(w)$ shows that $S(w) = T^*(w) \ \forall w$

$\implies S = T^*$. $\square$

<u>Ex:</u> $T: \mathbb{C}^3 \to \mathbb{C}^2, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} y-3iz \\ (1+i)x \end{pmatrix}$

Choose standard inner products on $\mathbb{C}^3$ and $\mathbb{C}^2$ (e.g. $\langle v,w\rangle_{\mathbb{C}^2} = v_1\overline{w_1} + v_2\overline{w_2}$)

Let $\begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{C}^2, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{C}^3$ be arbitrary:

$$\left\langle T\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, T^*\begin{pmatrix} p \\ q \end{pmatrix} \right\rangle$$

$$\begin{aligned} LHS = \left\langle \begin{pmatrix} y-3iz \\ (1+i)x \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix} \right\rangle &= (y-3iz)\cdot\overline{p} + (1+i)x\cdot\overline{q} \\ &= x\cdot \underbrace{(1+i)\overline{q}}_{\overline{(1-i)q}} + y\cdot\overline{p} + z\cdot\underbrace{(-3i)\overline{p}}_{\overline{3ip}} \\ &= \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} (1-i)q \\ p \\ 3ip \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, T^*\begin{pmatrix} p \\ q \end{pmatrix} \right\rangle \end{aligned}$$

$\Rightarrow T^*: \mathbb{C}^2 \to \mathbb{C}^3, \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} (1-i)q \\ p \\ 3ip \end{pmatrix}$

Let's compute the matrix representations of T and T^* (w.r.t. stand. basis):

$$A = \mathcal{M}(T) = \begin{pmatrix} 0 & 1 & -3i \\ 1+i & 0 & 0 \end{pmatrix}, \quad B = \mathcal{M}(T^*) = \begin{pmatrix} 0 & 1-i \\ 1 & 0 \\ 3i & 0 \end{pmatrix}$$

$\Rightarrow B = \left(\underset{\substack{\uparrow \\ \text{entry-wise complex conjugate}}}{\overline{A}}\right)^T =: A^*$ ... conjugate transpose of A.

Prop Let $T \in \mathcal{L}_{\mathbb{F}}(V,W)$, and $B_V = \{v_1, \dots, v_n\}$ and $B_W = \{w_1, \dots, w_m\}$ be orthonormal bases for V and W, resp.

Then,
$$\mathcal{M}(T^*)_{B_W, B_V} = \left(\mathcal{M}(T)_{B_V, B_W}\right)^*$$

__Proof:__ HW

Prop 7.6 Properties of the adjoint

i) $(S+T)^* = S^* + T^*$ for all $S, T \in \mathcal{L}(V,W)$

ii) $(\lambda T)^* = \bar{\lambda}\, T^*$ for all $T \in \mathcal{L}(V,W)$, $\lambda \in \mathbb{C}$

iii) $(T^*)^* = T$ for all $T \in \mathcal{L}(V,W)$

iv) $I_V^* = I_V$

v) $(ST)^* = T^* S^*$ for all $T \in \mathcal{L}(V,W)$, $S(W,U)$

The same relations hold for matrices and the conjugate transpose

$$A^* = (\bar{A})^T.$$

__Proof:__ $v \in V$, $w \in W$ arbitrary.

i)
$$\begin{aligned}
\langle v, (S+T)^*(w) \rangle &= \langle (S+T)(v), w \rangle \\
&= \langle S(v) + T(v), w \rangle \\
&= \langle S(v), w \rangle + \langle T(v), w \rangle \\
&= \langle v, S^*(w) \rangle + \langle v, T^*(w) \rangle = \langle v, \overbrace{S^*(w) + T^*(w)}^{(S^* + T^*)(w)} \rangle
\end{aligned}$$

ii) similar (exercise).

iii) $\langle w, (T^*)^*(v) \rangle = \langle T^*(w), v \rangle = \overline{\langle v, T^*(w) \rangle}$

$= \overline{\langle T(v), w \rangle} = \langle w, T(v) \rangle$

$\Rightarrow T = (T^*)^*.$

iv) $v, \tilde{v} \in V$: $\langle v, \tilde{v} \rangle = \langle v, I_V(\tilde{v}) \rangle = \langle I_V(v), \tilde{v} \rangle$

v) $T \in \mathcal{L}(V,W)$, $S \in \mathcal{L}(W,U)$: $ST \in \mathcal{L}(V,U)$

$(ST)^* : U \to V$, for $u \in U$:

$\langle v, (ST)^*(u) \rangle = \langle (ST)(v), u \rangle = \langle S(T(v)), u \rangle$

$= \langle T(v), S^*(u) \rangle = \langle v, T^*(S^*(u)) \rangle$

$= \langle v, (T^* S^*)(u) \rangle$

$\Rightarrow (ST)^* = T^* S^*$ $\square$

$\boxed{\text{Prop 7.7}}$ Let $T \in \mathcal{L}(V,W)$.

i) $\ker T^* = (\operatorname{im} T)^\perp$

ii) $\operatorname{im} T^* = (\ker T)^\perp$

iii) $\ker T = (\operatorname{im} T^*)^\perp$

v) $\operatorname{im} T = (\ker T^*)^\perp$

Proof: i) $w \in \ker T^* \iff T^*(w) = 0$

$\iff \langle v, T^*(w) \rangle = 0 \quad \forall v \in V$

$\iff \langle T(v), w \rangle = 0 \quad \forall v \in V$

$\iff \langle \tilde{w}, w \rangle = 0 \quad \forall \tilde{w} \in \operatorname{im} T$

$\iff w \in (\operatorname{im} T)^\perp$

iv) $\ker T^* = (\operatorname{im} T)^\perp \implies (\ker T^*)^\perp = ((\operatorname{im} T)^\perp)^\perp = \operatorname{im} T$

by Prop 6.51.

ii), iii) replace T by T^* and use i), iv) with $T = (T^*)^*$. $\square$