Lecture 23: Inner product spaces

Last time: Eigenspaces and diagonalization

Recall: Dot product (a scalar product) in $\mathbb{R}^n$

$$x, y \in \mathbb{R}^n, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad x_i, y_i \in \mathbb{R}$$

then $\quad x \cdot y = x_1 \cdot y_1 + \ldots + x_n \cdot y_n = \sum_{i=1}^{n} x_i y_i$

Geometrically, dot product allows us to talk about angles between vectors and the length of a vector (and also the distance between points).

Some properties:
- $x \cdot x \geq 0 \quad \forall x$, and $x \cdot x = 0$ iff $x = 0$.
- $x \cdot y = y \cdot x \quad \forall x, y \in \mathbb{R}^n$
- $x \cdot (\lambda_1 y_1 + \lambda_2 y_2) = \lambda_1 x \cdot y_1 + \lambda_2 x \cdot y_2$

  for $x, y_1, y_2 \in \mathbb{R}^n$, $\lambda_1, \lambda_2 \in \mathbb{R}$.

We now generalize the (real) dot product to arbitrary complex VS's:

$\boxed{\text{Def 6.3}}$ Inner product on complex vector spaces

Let V be a VS over $\mathbb{C}$. A function $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{C}$ is called an inner product, if it satisfies the following properties:

i) Positive definiteness: $\langle v, v \rangle \geq 0 \quad \forall v \in V$

(in particular, $\langle v, v \rangle \in \mathbb{R}$)

and $\langle v, v \rangle = 0$ if and only if $v = 0$.

ii) Linearity in first argument:

$$\langle \lambda_1 u_1 + \lambda_2 u_2 , v \rangle = \lambda_1 \langle u_1, v \rangle + \lambda_2 \langle u_2, v \rangle$$

for all $u_1, u_2, v \in V$, $\lambda_1, \lambda_2 \in \mathbb{C}$.

iii) Conjugate symmetry: $\langle u, v \rangle = \overline{\langle v, u \rangle} \quad \forall u, v \in V$

Remark: ii) + iii) imply "conjugate" linearity in second argument:

$$\langle v, \lambda_1 u_1 + \lambda_2 u_2 \rangle = \overline{\lambda_1} \langle v, u_1 \rangle + \overline{\lambda_2} \langle v, u_2 \rangle$$

$$\forall v, u_1, u_2 \in V, \ \lambda_1, \lambda_2 \in \mathbb{C}.$$

because:
$$\begin{aligned}
\langle v, \lambda_1 u_1 + \lambda_2 u_2 \rangle &\overset{iii)}{=} \overline{\langle \lambda_1 u_1 + \lambda_2 u_2 , v \rangle} \\
&\overset{(ii)}{=} \overline{\lambda_1 \langle u_1, v \rangle + \lambda_2 \langle u_2, v \rangle} \\
&= \overline{\lambda_1}\, \overline{\langle u_1, v \rangle} + \overline{\lambda_2}\, \overline{\langle u_2, v \rangle} \\
&\overset{(iii)}{=} \overline{\lambda_1} \langle v, u_1 \rangle + \overline{\lambda_2} \langle v, u_2 \rangle \qquad \square
\end{aligned}$$

Ex.: ·) Euclidean inner product on $\mathbb{C}^n$:

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad x_i, y_i \in \mathbb{C},$$

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \cdot \overline{y_i} = x_1 \overline{y_1} + \dots + x_n \overline{y_n}$$

(for $\mathbb{R}^n$, this coincides with the dot product)

·) More generally, let $c_i \in \mathbb{R}$, $c_i > 0$, $i = 1, \dots, n$,

then $\langle x, y \rangle_c := \sum_{i=1}^{n} c_i \, x_i \, \overline{y_i}$ is an inner product on $\mathbb{C}^n$ (HW)

·) Let $V = C([a,b])$, the space of continuous real-valued functions defined on $[a,b]$ where $a < b$.

$f, g \in V$: $\langle f, g \rangle := \int_a^b f(x) \, g(x) \, dx$ is an inner product (HW)

$\boxed{\text{Def 6.5}}$ An inner product space is a complex vector space V with an inner product $\langle \cdot, \cdot \rangle$ on V.

Remark: It follows from the definitions, that

i) $v \mapsto \langle v, u \rangle$ is a linear map $V \to \mathbb{C}$ for every fixed $u \in V$.

ii) $\langle 0, v \rangle = \langle v, 0 \rangle = 0 \quad \forall v \in V$

$$\left(\langle 0, v \rangle = \langle 0+0, v \rangle = \langle 0, v \rangle + \langle 0, v \rangle \Rightarrow \langle 0, v \rangle = 0 \right.$$
$$\left. \to \langle v, 0 \rangle = 0 \right)$$

iii) Conjugate linearity in the second argument:

$$\langle v, \lambda_1 u_1 + \lambda_2 u_2 \rangle = \bar{\lambda}_1 \langle v, u_1 \rangle + \bar{\lambda}_2 \langle v, u_2 \rangle$$

$$\forall\ v, u_1, u_2 \in V, \quad \lambda_1, \lambda_2 \in \mathbb{C}.$$

Norms

Recall the vector norm $\|.\|$ on $\mathbb{R}^n$: $\|x\| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$

This is related to the dot product on $\mathbb{R}^n$: $\|x\| = \sqrt{x \cdot x}$

More generally:

(for $\lambda \in \mathbb{C}$, $\lambda = a + bi$, $a, b \in \mathbb{R}$:
$|\lambda| = \sqrt{a^2+b^2} = \sqrt{\lambda \cdot \bar{\lambda}}$)

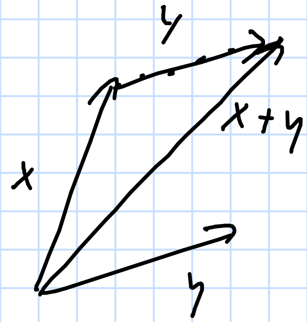
Def Norms

Let V be a complex VS. A function $\|.\| : V \to \mathbb{R}$ is called a norm if it satisfies the following properties:

i) Positive definiteness: $\|x\| = 0$ iff $x = 0 \in V$.

ii) Absolute homogeneity: $\|\lambda x\| = |\lambda| \cdot \|x\|$ for $x \in V$, $\lambda \in \mathbb{C}$

iii) Triangle inequality: $\|x+y\| \le \|x\| + \|y\|$ for $x, y \in V$.



triangle inequality: $\|x+y\| \leq \|x\| + \|y\|$

Note that by Properties i), ii), iii), we have $\|x\| \geq 0 \quad \forall x \in V$:

$$0 \overset{(i)}{=} \|0\| = \|x + (-x)\| \overset{(iii)}{\leq} \|x\| + \|-x\|$$

$$\overset{(ii)}{=} \|x\| + |-1| \cdot \|x\|$$

$$= 2\|x\|$$

$\boxed{\text{Prop}}$ Let V be an inner product space with inner product $\langle \cdot , \cdot \rangle$

Then $\|x\| := \sqrt{\langle x, x \rangle}$ (positive square root)

defines a norm on V.

Proof: Need to check Properties i), ii), iii) in the definition above.

i) $\|x\| = 0 = \sqrt{\langle x, x \rangle}$ iff $0 = \langle x, x \rangle$ iff $x = 0$.
(inner product).

ii) $\lambda \in \mathbb{C}$, $x \in V$: $\underline{\|\lambda x\|^2} = \langle \lambda x, \lambda x \rangle = \lambda \langle x, \lambda x \rangle$

$$= \lambda \cdot \bar{\lambda} \langle x, x \rangle = \underline{|\lambda|^2 \|x\|^2}$$

iii) will be proved in the next lecture!