Lecture 19: Rank of a matrix

Last time: Quotient spaces

$\boxed{\text{Def}}$ Let $A \in M_{m \times n}(\mathbb{F})$, and let $a_j \in \mathbb{F}^m$ be the j-th column of A, $1 \leq j \leq n$, i.e., $A = (a_1 | \dots | a_j | \dots | a_n)$

Then the column rank of A is the dimension of the space spanned by its columns:

$$\text{col rk } A = \dim \langle a_1, \dots, a_n \rangle \quad (a_j \in \mathbb{F}^m)$$

Let $\tilde{a}_i \in \mathbb{F}^n$ be the i-th row of A, $1 \leq i \leq m$, i.e., $A = \begin{pmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_i \\ \vdots \\ \tilde{a}_m \end{pmatrix}$

Then the row rank of A is the dimension of the space spanned by its rows:

$$\text{row rk } A = \dim \langle \tilde{a}_1, \dots, \tilde{a}_m \rangle \quad (\tilde{a}_i \in \mathbb{F}^n)$$

Easy to see: $T \in \mathcal{L}_{\mathbb{F}}(V, W)$, $\dim V = n$, $\dim W = m$,

$$\mathcal{M}(T) = A \in M_{m,n}(\mathbb{F}), \text{ then } \text{col rk } A = \dim \operatorname{im} T.$$

$\boxed{\text{Def}}$ transpose of a matrix

Let $A \in M_{m,n}(\mathbb{F})$ with components $A_{ij} \in \mathbb{F}$, $1 \leq i \leq m$, $1 \leq j \leq n$.

Then the transpose of A, denoted A^T, is the matrix of size $n \times m$ with components $(A^T)_{ij} = A_{ji}$.

Ex.: $A = \begin{pmatrix} 3 & -1 & 4 \\ 2 & 2 & 1 \end{pmatrix} \in M_{2,3}(\mathbb{R})$

$\Rightarrow A^T = \begin{pmatrix} 3 & 2 \\ -1 & 2 \\ 4 & 1 \end{pmatrix} \in M_{3,2}(\mathbb{R})$

Prop colrk A = rowrk A^T, rowrk A = colrk A^T

Proof: Follows from the definitions.

Prop Rank of a matrix

For any $A \in M_{m,n}(\mathbb{F})$, colrk A = rowrk A.

This number is simply called the rank of A, denoted rk A.

Proof: Let colrk $A = r$ $(r \leq n)$. $A = (a_1 | \dots | a_n)$, $a_j \in \mathbb{F}^m$

By def., the span $\langle a_1, \dots, a_n \rangle \leq \mathbb{F}^m$ has dim. r.

Let $\{c_1, \dots, c_r\}$ be a basis for $\langle a_1, \dots, a_n \rangle$

Then we can express the columns a_j of A as unique linear combinations of these basis vectors:

$\forall j = 1, \dots, n: \quad a_j = \sum_{k=1}^{r} \lambda_{kj} c_k \quad$ for $\lambda_{kj} \in \mathbb{F}$

Define a matrix $C \in M_{m,r}(\mathbb{F})$, $C = (c_1 | \dots | c_r)$, i.e., $C_{jk} = (c_k)_j$

and a matrix $R \in M_{r,n}(\mathbb{F})$, $R_{kj} = \lambda_{kj}$

$$a_j = \sum_{k=1}^{r} \lambda_{kj} c_k \iff A = C \cdot R$$

$$\iff A_{ij} = \sum_{k=1}^{r} C_{ik} R_{kj} \qquad (*)$$

Let $\tilde{a}_i \in \mathbb{F}^n$ be the i-th row of A, i.e., $A = \begin{pmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_m \end{pmatrix}$

$(\tilde{a}_i)_j = A_{ij}$

$R \in M_{r,n}(\mathbb{F})$, $R = \begin{pmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_r \end{pmatrix}$

Then $(*)$ can be rewritten as $\tilde{a}_i = \sum_{k=1}^{r} C_{ik} \tilde{b}_k$, where

$\tilde{b}_k \in \mathbb{F}^n$ is the k-th row of the matrix R, $1 \le k \le r$.

$\Rightarrow \langle \tilde{a}_1, \dots, \tilde{a}_m \rangle \subseteq \langle \tilde{b}_1, \dots, \tilde{b}_r \rangle$

$\Rightarrow$ row rk $A = \dim \langle \tilde{a}_1, \dots, \tilde{a}_m \rangle \le r =$ col rk A.

Since A was arbitrary, we can apply the same argument to the transpose A^T of A: row rk $A^T \le$ col rk A^T.

Hence, row rk $A \le$ col rk $A =$ row rk $A^T \le$ col rk $A^T =$ row rk A,

so that row rk $A =$ col rk A. $\square$

$\boxed{\text{Cor}}$ Let $A \in M_n(\mathbb{F})$, then A is invertible iff rk $A = n$.

Proof: A is invertible $\iff$ columns of A (as vec's in $\mathbb{F}^n$) are lin. indep.

$\iff$ n columns of A form a basis of $\mathbb{F}^n$ $\iff$ rk $A = n$.

$\square$