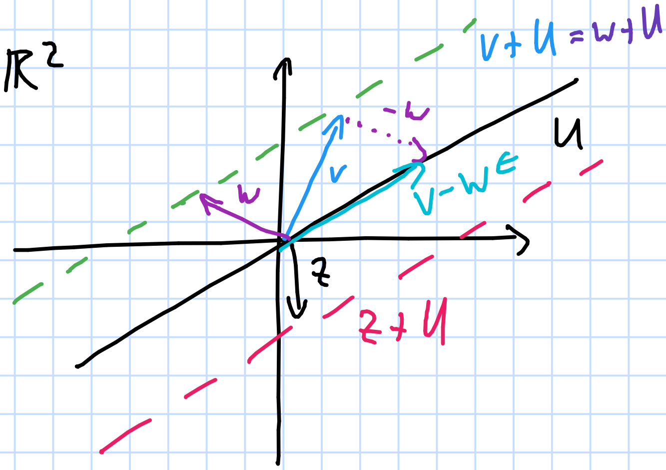
Lecture 18: Quotient spaces

Last time: Invertibility and basis change

Let V be a vector space over $\mathbb{F}$ (finite-dim.), and $U \leq V$ a subspace.

Idea: Partition V w.r.t. the subspace U



$\boxed{\text{Def}}$ Define an equivalence relation $\sim_U$ on vectors in V:

$$v \sim_U w \;:\Leftrightarrow\; v-w \in U$$

This is indeed an equivalence relation:

- ·) reflexive: $\forall v \in V$, $v \sim_U v$ (because $v - v = 0 \in U$)
- ·) symmetric: $\forall v, w \in V$: $v \sim_U w \Leftrightarrow w \sim_U v$ (if $v - w \in U$, then also $(-1)(v-w) = w - v \in U$)
- ·) transitive: $\forall v, w, z \in V$: if $v \sim_U w$, $w \sim_U z$, then $v \sim_U z$ (because if $v - w \in U$ and $w - z \in U$, then $v - w + w - z = v - z \in U$)

The equivalence classes are given by $(v \in V)$

$$v + U := \{ v + u : u \in U \} = [v]_U$$

$\boxed{\text{Prop 3.85}}$ Let $U \subseteq V$ and $v, w \in V$. TFAE:

i) $v - w \in U$ ii) $v + U = w + U$ iii) $(v+U) \cap (w+U) \neq \emptyset$

<u>Proof:</u> i) $\Rightarrow$ ii) Let $u \in U$, then

$$v + u = w + (\underbrace{v-w}_{\in U} + \underbrace{u}_{\in U}) = w + u' \quad \text{with } u' = v - w + u \in U$$

$\Rightarrow v + U \subseteq w + U$, and similarly $w + U \subseteq v + U$

$\Rightarrow v + U = w + U$.

ii) $\Rightarrow$ iii) if $v + U = w + U$, then $(v+U) \cap (w+U)$ is clearly non-empty.

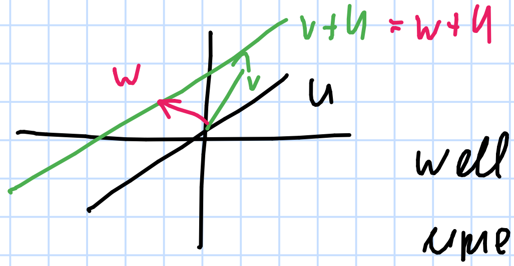
iii) $\Rightarrow$ i) if $(v+U) \cap (w+U) \neq \emptyset$, then there exist $u_1, u_2 \in U$

s.t. $v + u_1 = w + u_2 \Rightarrow v - w = u_2 - u_1 \in U$ $\square$

We want to define a vector space structure on the set of equivalence classes $\{v + U : v \in V\}$:

·) addition: let $v, w \in V$, then $(v+U) + (w+U) := (v+w) + U$

·) scalar multiplication: Let $v \in V$, $\lambda \in \mathbb{F}$, then $\lambda(v+U) := (\lambda v) + U$



Need to make sure that these definitions are well-defined, i.e., do <u>not</u> depend on the chosen representatives!

·) Let $v, w, \tilde{v}, \tilde{w} \in V$ s.t. $v + U = \tilde{v} + U$, and $w + U = \tilde{w} + U$

Then by Prop 3.85, $v - \tilde{v} \in U$, $w - \tilde{w} \in U$

But U is a subspace, so $v - \tilde{v} + w - \tilde{w} \in U$

$$\begin{gathered} v - \tilde{v} + w - \tilde{w} \\ \parallel \\ v + w - (\tilde{v} + \tilde{w}) \end{gathered}$$

$\Rightarrow v + w + U = \tilde{v} + \tilde{w} + U$ again by Prop 3.85.

·) Proof for scalar multiplication is similar.

$\boxed{\text{Prop } 3.87}$ Quotient space

Let $U \leq V$ be a subspace. Then the quotient space

$$V/U := \{ v + U : v \in V \}$$

is a vector space over $\mathbb{F}$ w.r.t. addition and scalar mult. defined above.

Proof: Exercise. (additive neutral element: $0 + U = U$). $\square$

$\boxed{\text{Prop } 3.89}$ Dimension of a quotient space

Let $U \leq V$, then $\dim V/U = \dim V - \dim U$

Proof: Define the "quotient map" $\pi : V \to V/U$,

$$\pi(v) = v + U.$$

·) image: clearly, $\operatorname{im} \pi = V/U$, because for all $v + U \in V/U$ there exists $v \in V$ (by def.) s.t. $\pi(v) = v + U$.

·) kernel: $\underbrace{\pi(v)}_{= \underline{v+U}} = 0_{V/U} = \underline{0 + U} = U$

By Prop. 3.85: $v + U = 0 + U \iff v \in U$

Hence, $\ker \pi = U$

By Prop 3.22, $\dim V = \dim \ker \pi + \dim \operatorname{im} \pi$

$$= \dim U + \dim V/U \qquad \square$$

Let $T \in \mathcal{L}_{\mathbb{F}}(V, W)$, and define a map ($\ker T \leq V$)

$$\tilde{T} : V/\ker T \to W$$
$$\tilde{T}(v + \ker T) = T(v)$$

$\boxed{\text{Prop } 3.91}$ i) $\tilde{T} \in \mathcal{L}_{\mathbb{F}}(V/\ker T, W)$

ii) $\ker \tilde{T} = \{0 + \ker T\} = \{0_{V/\ker T}\}$, i.e., $\tilde{T}$ is injective

iii) $\operatorname{im} \tilde{T} = \operatorname{im} T$, and hence $\tilde{T} : V/\ker T \to \operatorname{im} T$ is surjective.

iv) $V/\ker T \cong \operatorname{im} T$

<u>Proof</u>: i) $\tilde{T} : V/\ker T \to W$, $\tilde{T}(v + \ker T) = T(v)$

let $v_1 + \ker T = v_2 + \ker T$, then $v_1 - v_2 \in \ker T$ by Prop 3.85

$\Rightarrow T(v_1 - v_2) = 0 \Rightarrow T(v_1) = T(v_2)$.

$\tilde{T}$ is linear: simple exercise.

ii) Let $v \in V$ be s.t. $\tilde{T}(v + \ker T) = 0 = T(v)$

$\Rightarrow v \in \ker T$, and hence $v + \ker T = 0 + \ker T$ by Prop 3.85

$\Rightarrow \ker \tilde{T} = \{0 + \ker T\} = \{0_{V/\ker T}\}$.

iii) $\operatorname{im} \tilde{T} = \operatorname{im} T$ is clear by definition $(\tilde{T}(v + \ker T) = T(v))$

iv) $\tilde{T} : V/\ker T \to \operatorname{im} T$ is both injective (by ii)) and surjective (by iii))

$\Rightarrow \tilde{T}$ is invertible, hence an isomorphism. $\square$

<u>Remark</u>: Fundamental thm. of linear algebra:

If $T \in \mathcal{L}_{\mathbb{F}}(V, W)$, then $\dim V = \dim \ker T + \dim \operatorname{im} T$

Here: $V/\ker T \cong \operatorname{im} T \Rightarrow \dim V/\ker T \overset{\text{Prop 3.85}}{=} \dim V - \dim \ker T$

$= \dim \operatorname{im} T$.

Recall: $\ker T \leq V$, there exists another $X \leq V$ s.t.

$$V = \ker T \oplus X$$

We have that $\operatorname{im} T \cong X$ (Ex.).