Lecture 15: Invertibility of matrices and elementary matrices

Last time: Matrix multiplication

$A \in M_{m,n}(\mathbb{F}),\ B \in M_{n,p}(\mathbb{F}) \Rightarrow A \cdot B \in M_{m,p}(\mathbb{F})$

$$(A\cdot B)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}, \quad 1 \leq i \leq m, \ 1 \leq j \leq p$$

Prop Properties of the matrix product

- $A \in M_{m,n}(\mathbb{F}),\ B \in M_{n,p}(\mathbb{F}),\ C \in M_{p,q}(\mathbb{F})$:

$$A \cdot (B \cdot C) = (A \cdot B) \cdot C$$

- Identity matrix $I_n \in M_{n,n}(\mathbb{F})$, $I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$ is the neutral element for matrix mult: $A \in M_{m,n}(\mathbb{F})$:

$$I_m \cdot A = A = A \cdot I_n$$

- $A, D \in M_{m,n}(\mathbb{F}),\ B, C \in M_{n,p}(\mathbb{F})$:

$$A \cdot (B + C) = AB + AC$$

$$(A + D) \cdot B = AB + DB$$

Proof: Easy to verify. $\square$

Now: restrict to square matrices, where the numbers of rows and columns are equal. Notation: $M_n(\mathbb{F}) \equiv M_{n,n}(\mathbb{F})$

Def A matrix $A \in M_n(\mathbb{F})$ is called invertible if there exists another matrix $B \in M_n(\mathbb{F})$ with $A \cdot B = I_n = B \cdot A$.

<u>Ex.:</u> $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, then $B = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$ satisfies

$$A \cdot B = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B \cdot A$$

<u>Ex.:</u> $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ is not invertible!

Prop Let $A \in M_n(\mathbb{F})$. If A is invertible, there exists a <u>unique</u> $B \in M_n(\mathbb{F})$ s.t. $AB = I_n = BA$, denoted by A^{-1} and called the inverse of A.

<u>Proof:</u> Let $B_1, B_2 \in M_n(\mathbb{F})$ be such that $B_i A = I_n = A B_i$ for $i = 1, 2$.

Then: $\underline{B_1} = B_1 \cdot I_n = B_1 (A \cdot B_2) = (B_1 A) B_2 = I_n \cdot B_2 = \underline{B_2}$. $\square$

Prop Properties of inverses

i) If $A \in M_n(\mathbb{F})$ is invertible, then so is A^{-1}, and $(A^{-1})^{-1} = A$.

ii) If $A, B \in M_n(\mathbb{F})$ are invertible, then $A \cdot B$ is invertible, and $(AB)^{-1} = B^{-1} \cdot A^{-1}$

iii) If $A \in M_n(\mathbb{F})$ is invertible, then the system of linear eq's $Ax = b$ with $x, b \in \mathbb{F}^n$ has the unique solution $x = A^{-1} b$.

Proof: i) A invertible: $A \cdot A^{-1} = I_n = A^{-1}A \Rightarrow (A^{-1})^{-1} = A$

ii) A, B invertible: $A\underbrace{B B^{-1}}A^{-1} = A I_n A^{-1} = AA^{-1} = I_n = B^{-1}A^{-1}AB$

$$\Rightarrow (AB)^{-1} = B^{-1}A^{-1}.$$

iii) A invertible, A $n \times n$ matrix: $Ax = b$, $x, b \in \mathbb{F}^n$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : \quad Ax = \begin{pmatrix} A_{11}x_1 + \dots + A_{1n}x_n \\ \vdots \\ A_{n1}x_1 + \dots + A_{nn}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

multiply both sides by A^{-1}: $\underbrace{A^{-1}A}_{I_n} x = A^{-1}b$

$$x = \underline{A^{-1}b} \in \mathbb{F}^n \qquad \square$$

Elementary matrices

Idea: Write row operations on a matrix as multiplication by suitable matrices.

$\cdot$) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ 3a+g & 3b+h & 3c+i \end{pmatrix}$

$R3 \leftarrow R3 + 3R1$

$\cdot$) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{pmatrix}$ $\quad R2 \leftarrow 2R2$

·) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ g & h & i \\ d & e & f \end{pmatrix}$ $\quad R2 \leftrightarrow R3$

Def Elementary matrices

Let $1 \leq i, j \leq n$, $i \neq j$, and define matrices $P_{c,i,j}$, $Q_{c,i}$, $R_{i,j}$, $c \in \mathbb{F}$:

·) $P_{c,i,j} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & c & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \leftarrow$ i-th row $\quad$ "add c times the j-th row to the i-th row"

$\uparrow$ j-th col

·) $c \neq 0$: $\quad Q_{c,i} = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & c & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \leftarrow$ i-th row $\quad$ "multiply i-th row by c"

$\uparrow$ i-th col.

·) $R_{i,j} = \begin{pmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix} \begin{matrix} \leftarrow \text{i-th} \\ \leftarrow \text{j-th} \end{matrix}$ row $\quad$ "swap i-th and j-th row"

$\uparrow$ i-th $\quad \uparrow$ j-th col.

Prop Elementary matrices are invertible:

$$P_{c,i,j}^{-1} = P_{-c,i,j}, \quad Q_{c,i}^{-1} = Q_{1/c,i}, \quad R_{i,j}^{-1} = R_{i,j}$$

$\uparrow$ $c \neq 0$

$\boxed{\text{Prop}}$ A matrix $A \in M_n(\mathbb{F})$ is invertible if and only if it can be written as a product of elementary matrices.

Proof: ($\Leftarrow$) Assume $A = E_m \cdot \ldots \cdot E_1$, E_i some elementary matrices.

Then A is invertible as a product of invertible matrices,

$$A^{-1} = (E_m \cdot \ldots \cdot E_1)^{-1} = E_1^{-1} \cdot \ldots \cdot E_m^{-1}.$$

($\Rightarrow$) Let A be invertible, then the SLE $Ax = b$ $(x, b \in \mathbb{F}^n)$ has a unique solution $x = A^{-1}b \iff \underline{I_n \cdot x = A^{-1}b \in \mathbb{F}^n}$

RREF of $I_n \cdot x = A^{-1}b$:

$$\left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \\ 0 & \ddots & & \vdots & \\ \vdots & & \ddots & 0 & A^{-1}b \\ 0 & \cdots & 0 & 1 & \end{array}\right)$$

which can be obtained from the aug. matrix of coefficients $(A \mid b)$ using a series of row operations. But each row operation corresponds to multiplying from the left by some elementary matrix E_i in the i-th step.

$$\left(\begin{array}{ccc|c} A_{11} & \cdots & A_{1n} & b_1 \\ \vdots & & & \vdots \\ A_{n1} & \cdots & A_{nn} & b_n \end{array}\right) \xrightarrow{\text{apply } E_1, E_2, \ldots, E_m} \left(\begin{array}{c|c} I_n & A^{-1}b \end{array}\right)$$

$$\underbrace{\left(\begin{array}{ccc|c} A_{11} & \cdots & A_{1n} & b_1 \\ \vdots & & & \vdots \\ A_{n1} & \cdots & A_{nn} & b_n \end{array}\right)}_{A} \xrightarrow{\text{apply } E_1, E_2, \dots, E_m} \left(\begin{array}{c|c} I_n & A^{-1}b \end{array}\right)$$

$$\Rightarrow A \xrightarrow{\text{row op's}} E_m \cdot \ldots \cdot E_1 \cdot A = I_n$$

$$\Leftrightarrow A = (E_m \cdot \ldots \cdot E_1)^{-1} = E_1^{-1} \cdot \ldots \cdot E_m^{-1}$$

$$= F_1 \cdot \ldots \cdot F_m$$

"(elementary matrix)$^{-1}$ = elem. matrix" $\square$

$\rightarrow$ Algorithm to check whether $A \in M_n(\mathbb{F})$ is invertible, and if yes, determine A^{-1}:

Write $M = \left(\begin{array}{c|c} \underline{A} & I_n \end{array}\right) \in M_{n,2n}(\mathbb{F})$

and bring M into RREF: $\left(\begin{array}{c|c} A' & B \end{array}\right)$

Then A is invertible with inverse $A^{-1} = B$ iff $A' = I_n$.

<u>Proof idea:</u>

if A is invertible, then $A = E_m \cdot \ldots \cdot E_1$, $A^{-1} = (E_m \cdot \ldots \cdot E_1)^{-1}$

$$\underline{= E_1^{-1} \cdot \ldots \cdot E_m^{-1}}$$

$$\Rightarrow E_1^{-1} \cdot \ldots E_m^{-1} A = I_n$$