Lecture 14: Matrix multiplication

Last time: Linear maps as matrices

Let $T \in \mathcal{L}_{\mathbb{F}}(V, W)$, fix bases $\mathcal{B}_V = \{v_1, \dots, v_n\}$ for V,

$\mathcal{B}_W = \{w_1, \dots, w_m\}$ for W.

Define $A_{ij} \in \mathbb{F}$, $1 \leq i \leq m$, $1 \leq j \leq n$, through

$$T(v_j) = A_{1j} w_1 + \dots + A_{mj} w_m, \quad 1 \leq j \leq n.$$

$$\Rightarrow \text{matrix } \mathcal{M}(T)_{\mathcal{B}_V, \mathcal{B}_W} = A = \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \dots & A_{mn} \end{pmatrix}$$

For $S, T \in \mathcal{L}_{\mathbb{F}}(V, W)$, $a \in \mathbb{F}$:

$$\left[\begin{array}{l} (S+T)(v) = S(v) + T(v) \\ (aT)(v) = aT(v) \end{array} \right.$$

- $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$
- $\mathcal{M}(aT) = a\,\mathcal{M}(T)$

$$\mathcal{L}_{\mathbb{F}}(V, W) \underset{(\cong)}{\longleftrightarrow} M_{m,n}(\mathbb{F}) \quad \text{for } m = \dim W, \ n = \dim V.$$

Let U, V, W be VS's over $\mathbb{F}$, $p = \dim U$, $n = \dim V$, $m = \dim W$

Let $T \in \mathcal{L}_{\mathbb{F}}(U, V)$, $S \in \mathcal{L}_{\mathbb{F}}(V, W)$, then $ST \in \mathcal{L}_{\mathbb{F}}(U, W)$,

$$ST(u) = S(T(u))$$

What matrix $D \in M_{m,p}(\mathbb{F})$ satisfies $D = \mathcal{M}(ST)$?

Let $B_U = \{u_1, \dots, u_p\}$, $B_V = \{v_1, \dots, v_n\}$, $B_W = \{w_1, \dots, w_m\}$ be bases for U, V, W, resp.

$T \in \mathcal{L}_{\mathbb{F}}(U, V)$, $S \in \mathcal{L}_{\mathbb{F}}(V, W)$, $A = \mathcal{M}(S)_{B_V, B_W}$, $C = \mathcal{M}(T)_{B_U, B_V}$

Then $T(u_k) = \sum_{j=1}^{n} C_{jk} v_j$ and $S(v_j) = \sum_{i=1}^{m} A_{ij} w_i$

(where $1 \le k \le p$ and $1 \le i \le m$)

Expand
$$
\begin{aligned}
ST(u_k) &= S(T(u_k)) \\
&= S\left(\sum_{j=1}^{n} C_{jk} v_j\right) \\
&= \sum_{j=1}^{n} C_{jk} S(v_j) \\
&= \sum_{j=1}^{n} C_{jk} \sum_{i=1}^{m} A_{ij} w_i \\
&= \sum_{i=1}^{m} \underbrace{\left(\sum_{j=1}^{n} A_{ij} C_{jk}\right)}_{=: D_{ik}} w_i = \sum_{i=1}^{m} D_{ik} w_i
\end{aligned}
$$

That is, $D = A \cdot C$ with $D_{ik} = \sum_{j=1}^{n} A_{ij} C_{jk}$, $1 \le i \le m$, $1 \le k \le p$

$D \in M_{m,p}(\mathbb{F})$, $\mathcal{M}(ST) = D = A \cdot C = \mathcal{M}(S) \cdot \mathcal{M}(T)$

$\boxed{\text{Def } 3.41}$ Let $A \in M_{m,n}(\mathbb{F})$, $C \in M_{n,p}(\mathbb{F})$,

then $D = A \cdot C \in M_{m,p}(\mathbb{F})$ with $D_{ik} = \sum_{j=1}^{n} A_{ij} C_{jk}$

"$(m \times n) \cdot (n \times p) = m \times p$"

<u>Ex.:</u> $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 6 & 5 & 4 & 3 \\ 2 & 1 & 0 & -1 \end{pmatrix}$

3×2 $\qquad$ 2×4

$(A \cdot B)_{2,3} = 3 \cdot 4 + 4 \cdot 0 = 12$

$$A \cdot B = \begin{pmatrix} 1 \cdot 6 + 2 \cdot 2 & & \\ 3 \cdot 6 + 4 \cdot 2 & \dots & \\ 5 \cdot 6 + 6 \cdot 2 & & \end{pmatrix} = \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 42 & 31 & 20 & 9 \end{pmatrix}$$

$B \cdot A$ is not defined!

<u>Ex.:</u> $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 4 \\ 2 & 2 \end{pmatrix}$: $\quad A \cdot B = \begin{pmatrix} 2 \cdot 1 + 3 \cdot 2 & 2 \cdot 4 + 3 \cdot 2 \\ 1 \cdot 1 + 2 \cdot 2 & 1 \cdot 4 + 2 \cdot 2 \end{pmatrix}$

$$= \begin{pmatrix} 8 & 14 \\ 5 & 8 \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} 6 & 11 \\ 6 & 10 \end{pmatrix} \neq A \cdot B$$

Linear maps acting by matrix multiplication

V, W VS's, $\dim V = n$, $\dim W = m$, $T \in \mathcal{L}_{\mathbb{F}}(V, W) \leftrightarrow A = \mathcal{M}(T)_{B_V, B_W} \in M_{m,n}(\mathbb{F})$

$\downarrow$ basis $B_V = \{v_1, \dots, v_n\}$ $\qquad$ $\rightarrow$ basis $B_W = \{w_1, \dots, w_m\}$

$B_V = \{v_1, \dots, v_n\}$ is a basis for V:

for all $v \in V$, there are c_j, $1 \leq j \leq n$ with

$$v = c_1 v_1 + \dots + c_n v_n \longleftrightarrow [v]_{B_V} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \mathcal{M}(v)_{B_V} \in M_{n,1}(\mathbb{F})$$

$\boxed{\text{Prop}}$ $$\mathcal{M}(T(v))_{B_W} = \mathcal{M}(T)_{B_V, B_W} \cdot \mathcal{M}(v)_{B_V}$$

$T \in \mathcal{L}_{\mathbb{F}}(V, W)$ $$= A \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

<u>Proof:</u> $\mathcal{M}(T(v))$: $T(v) = \sum_{i=1}^{m} d_i w_i$ for some $d_i \in \mathbb{F}$:

$$T(v) = T\left(\sum_{j=1}^{n} c_j v_j\right) = \sum_{j=1}^{n} c_j \underbrace{T(v_j)}_{= A_{1j} w_1 + \dots + A_{mj} w_m} = \sum_{j=1}^{n} c_j \sum_{i=1}^{m} A_{ij} w_i$$

$$= \sum_{i=1}^{m} \underbrace{\left(\sum_{j=1}^{n} A_{ij} c_j\right)}_{=: d_i} w_i$$

$$\Rightarrow d_i = \sum_{j=1}^{n} A_{ij} c_j = \left[A \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}\right]_i = [\mathcal{M}(T) \cdot \mathcal{M}(v)]_i$$

$$\Rightarrow \mathcal{M}(T(v)) = \mathcal{M}(T) \cdot \mathcal{M}(v) \qquad \square$$

Whenever we wrote $\mathbb{F}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{F} \right\}$, we implicitly

chose a basis $B_V = \{v_1, \dots, v_n\}$ s.t. $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \mathcal{M}(x)_{B_V}$.

E.g.: Let $B_1 = \{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$

and $v = e_1 + 2e_2 + 3e_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \mathcal{M}(v)_{B_1}$

Let $B_2 = \{f_1, f_2, f_3\}$, $f_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $f_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $f_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, be another basis for $\mathbb{R}^3$.

Then $v = -f_1 - f_2 + 3f_3 = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} = \mathcal{M}(v)_{B_2}$

Alternative way of understanding matrix multiplication:

$A \in M_{m,n}(\mathbb{F})$, $B = M_{n,p}(\mathbb{F}) \Rightarrow A \cdot B \in M_{m,p}(\mathbb{F})$

$$B = \begin{pmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{pmatrix} \longrightarrow A \cdot B = \begin{pmatrix} | & | & & | \\ A \cdot b_1 & A \cdot b_2 & \cdots & A b_p \\ | & | & & | \end{pmatrix}$$

b_k ... k-th column of B

$b_k \in M_{n,1}(\mathbb{F})$, $b_k = \begin{pmatrix} B_{1k} \\ \vdots \\ B_{nk} \end{pmatrix}$, then $(A \cdot b_k)_i = \sum_{j=1}^{n} A_{ij} (b_k)_j$

$$= \sum_{j=1}^{n} A_{ij} B_{jk} = (A \cdot B)_{i,k}$$