Lecture 9: Dimension of a vector space

Last time: Bases of vector spaces

Intuition: dimension $\hat{=}$ # degrees of freedom

E.g. $\mathbb{R}^3$ should have dim. 3

More generally, $\mathbb{F}^n$ should have dim. n for arbitrary $\mathbb{F}$.

Canonical choice of basis: Standard basis $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

$\rightarrow$ dimension $\hat{=}$ length of a basis

$\boxed{\text{Prop 2.35}}$ Any two bases of a finite-dim. vector space V have the same length.

Proof: Recall: $\mathcal{B} = \{v_1, \dots, v_n\}$ is a basis for V

$:\Longleftrightarrow \quad V = \langle v_1, \dots, v_n \rangle$ and $\{v_1, \dots, v_n\}$ is lin. indep.

Let $\mathcal{B}_1 = \{v_1, \dots, v_n\}$, $\mathcal{B}_2 = \{w_1, \dots, w_m\}$ be two bases for V.

Then $\mathcal{B}_1$ is lin. indep., $\mathcal{B}_2$ spans $V \Rightarrow n \leq m$ by Prop 2.23.

Reversing the roles of $\mathcal{B}_1$ and $\mathcal{B}_2$ also shows $m \leq n \Rightarrow m = n$. $\square$

Def. 2.36 Dimension of a vector space

The dimension of a vector space V (finite-dim.) is the length of any basis for V. We denote the dimension of V by $\dim V$. (By def., $\dim\{0\} = 0$; remember: $\langle \emptyset \rangle = \{0\}$)

Ex.: ·) $\dim \mathbb{F}^n = n$ for any field $\mathbb{F}$.

·) $\dim P_d(\mathbb{F}) = d+1$, since $\{1, x, \dots, x^d\}$ is a basis for $P_d(\mathbb{F})$. (HW3).

Prop (2.38, 2.39, 2.42)

Let V be a finite-dim. vector space.

i) If $U \leq V$ is a subspace of V, then $\dim U \leq \dim V$.

ii) Every lin. indep. set of vectors in V of length $\dim V$ is a basis for V.

iii) Every spanning set of vectors in V of length $\dim V$ is a basis for V.

Proof: i) Choose a basis $\{u_1, \dots, u_m\}$ of U s.t. $\dim U = m$.

In particular, the u_i are lin. indep. in U, and then in V.

$\Rightarrow \{u_1, \dots, u_m\}$ can be extended to a basis $\{u_1, \dots, u_m, v_1, \dots, v_k\}$ for V by Prop 2.33, s.t. $\dim V = m + k \geq m = \dim U$.

ii) By Prop 2.33, every set of lin. indep. vectors can be extended to a basis for V, but we already have $\dim V$ lin. indep. vectors, and by Prop 2.35 every basis has exactly $\dim V$ elements.

iii) By Prop 2.31 every spanning set for V contains a basis of length $\dim V$ (Prop 2.35) $\Rightarrow$ the $\dim V$ spanning vectors already form a basis for V. $\square$

Prop 2.43 Dimension of a sum of subspaces

Let $U_1, U_2 \leq V$ be subspaces of a finite-dim. vector space V.

Then $\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$

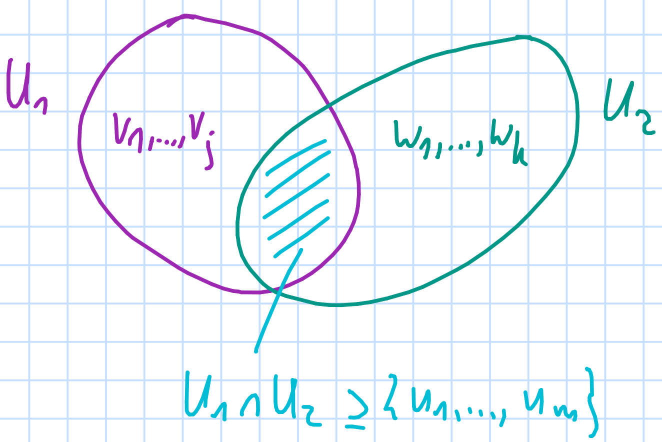
<u>Proof:</u> By Prop 1.34 (HW2), $U_1 \cap U_2$ is a subspace of V (also a subspace of U_1 and U_2).

Let $\{u_1, \dots, u_m\}$ be a basis for $U_1 \cap U_2$ s.t. $\dim(U_1 \cap U_2) = m$.

Since $\{u_1, \dots, u_m\}$ are lin. indep. in U_1, we can extend them to a basis $\{u_1, \dots, u_m, v_1, \dots, v_j\}$ for U_1 by Prop. 2.33.

$\Rightarrow \dim U_1 = m + j$.

By the same logic, we can extend $\{u_1, \dots, u_m\}$ to a basis $\{u_1, \dots, u_m, w_1, \dots, w_k\}$ for U_2 $\Rightarrow \dim U_2 = m + k$.



Claim: $\{u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k\}$ is a basis for $U_1 + U_2$.

If claim is true: $\dim(U_1 + U_2) = m + j + k$

$= m + j + m + k - m$

$= \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$

$\Rightarrow$ proves Prop.

Proof of the claim:

to show: a) $\{u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k\}$ span $U_1 + U_2$

b) — " — are linearly independent.

for a): clearly, $U_i \subseteq \langle u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k \rangle$ for $i = 1,2$

$\Rightarrow$ by Prop 1.39, $U_1 + U_2 \subseteq \langle u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k \rangle$

to show $\langle \dots \rangle \subseteq U_1 + U_2$, let $u \in \langle u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k \rangle$

then there are $a_i, b_\ell, c_n \in \mathbb{F}$ s.t.

$$u = \underbrace{\sum_{i=1}^{m} a_i u_i + \sum_{\ell=1}^{j} b_\ell v_\ell}_{\in U_1} + \underbrace{\sum_{n=1}^{k} c_n w_n}_{\in U_2} \in U_1 + U_2$$

$\Rightarrow \langle \dots \rangle \subseteq U_1 + U_2 \Rightarrow \langle \dots \rangle = U_1 + U_2. \Rightarrow$ a)

for b): Let $a_i, b_\ell, c_n \in \mathbb{F}$ be s.t.

$$\sum_{i=1}^{m} a_i u_i + \sum_{\ell=1}^{j} b_\ell v_\ell + \sum_{n=1}^{k} c_n w_n = 0 \qquad (*)$$

to show: $a_i = b_\ell = c_n = 0 \quad \forall i, \ell, n$

$$\overset{(*)}{\Rightarrow} \underbrace{\sum_n c_n w_n}_{\in U_2} = - \sum_i a_i u_i - \sum_\ell b_\ell v_\ell \in U_1$$

for some $d_i \in \mathbb{F}$

$$\Rightarrow \sum_n c_n w_n \in U_1 \cap U_2 \Rightarrow \sum_n c_n w_n = \sum_i d_i u_i \qquad (**)$$

but $\{u_1, \dots, u_m, w_1, \dots, w_k\}$ are lin. indep. (as a basis for U_2)

$\Rightarrow \quad c_1 = \ldots = c_n = 0 = d_1 = \ldots = d_m \quad$ in $(**)$

Now $(*)$ becomes $\quad \sum_i a_i u_i + \sum_l b_l v_l = 0$

But $\{u_1, \ldots, u_m, v_1, \ldots, v_j\}$ are lin. indep. (as a basis for U_1)

$\Rightarrow \quad a_1 = \ldots = a_m = 0 = b_1 = \ldots = b_j \quad$ in $(*)$.

$\Rightarrow \{u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_n\}$ are lin. indep. $\Rightarrow$ b) $\square$