# Lecture 6: Subspaces

Last time: vector spaces (definition, examples, properties)

**Def 1.32** A subset $U \subseteq V$ of a vector space $V$ is called a subspace, denoted by $U \leq V$, if $(U, +, \cdot)$ is a vector space, where $+$ and $\cdot$ are the operations in $V$ restricted to $U \subseteq V$.

**Examples:**

- $\left\{ \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} : x_1, x_2 \in \mathbb{F} \right\} \leq \mathbb{F}^3$
- Let $P_k(\mathbb{F})$ be the vector space of polynomials of degree $\leq k$ with coefficients in $\mathbb{F}$. Then $P_\ell(\mathbb{F}) \leq P_k(\mathbb{F})$ for all $0 \leq \ell \leq k$.
- Let $a_1, \ldots, a_n \in \mathbb{F}$ be fixed. Then the hyperplane defined by the linear equation $a_1 x_1 + \ldots + a_n x_n = 0$ is a subspace of $\mathbb{F}^n$:

$$\{ x \in \mathbb{F}^n : a_1 x_1 + \ldots + a_n x_n = 0 \} \leq \mathbb{F}^n.$$

**Prop 1.34** Conditions for a subspace

$U \subseteq V$ is a subspace of $V$ if and only if:

i) $0_V \in U$ $\quad (0_U = 0_V)$

ii) $u, v \in U \Rightarrow u + v \in U$

iii) $a \in \mathbb{F}, u \in U \Rightarrow au \in U$

**Proof:** ($\Rightarrow$) $U \leq V$ means that $U$ is a vector space

$\Rightarrow$ ii), iii)

for i) for any $u \in U$, $\underbrace{0}_{=0_V} = u + \underbrace{(-u)}_{\in U} \in U \Rightarrow$ by Prop 1.25, $0_U = 0_V$.

($\Leftarrow$) ii), iii) $\Rightarrow$ $U$ is closed under $+$ and $\cdot$.

(V3): Inverses first: $u \in U \Rightarrow -u = \underbrace{(-1)}_{\substack{\text{add. inverse of} \\ 1 \in \mathbb{F}}} u \in U$ by (iii)

$\Rightarrow$ every $u \in U$ has an additive inverse.

(V2): satisfied, because $0_V \in U$ by (i).

(V1), (V4), (S1), (S2), (D1), (D2) hold in $V$, and therefore

in particular for $U \subseteq V$. $\Rightarrow$ $U$ is a vector space by Def. 1.19

$\Rightarrow U \leq V$ $\square$

## Sums of vector spaces

$\boxed{\text{Def 1.36}}$ Let $V$ be a vector space, and $U_1, \dots, U_m \subseteq V$ subsets.

Then we define their sum as $U_1 + \dots + U_m = \{ u_1 + \dots + u_m : u_i \in U_i$ for $1 \leq i \leq m. \}$

$\boxed{\text{Prop 1.39}}$ Let $U_i \leq V$ be subspaces of $V$, $1 \leq i \leq m$, then also $\sum_{i=1}^{m} U_i \leq V$ is a subspace, and it is the smallest subspace of $V$ that contains all $U_i$'s as subspaces.

Proof: $\sum_{i=1}^{m} U_i$ is clearly closed under addition and scalar multiplication, since each $U_i$ is a subspace ($u, v \in U_i \Rightarrow u+v \in U_i$).
$a \in \mathbb{F}, u \in U_i \Rightarrow au \in U_i$

$\Rightarrow$ ii), iii) in Prop 1.34 are satisfied.

For i), note that $0 \in U_i$ for all $i = 1, \dots, m$, and hence

$$0 = 0 + \dots + 0 \in \sum_{i=1}^{m} U_i \qquad \Rightarrow \sum_{i=1}^{m} U_i \leq V \text{ by Prop 1.34.}$$

We have $U_j \leq \sum_{i=1}^{m} U_i$ for all $j$, because

$$U_j \ni u_j = 0 + \dots + 0 + u_j + 0 + \dots + 0 \in \sum_{i=1}^{m} U_i$$

Moreover, let $W \leq V$ such that $U_j \leq W$ for all $1 \leq j \leq m$.
Then $\sum_{i=1}^{m} U_i \leq W$, since $W$ is closed under addition as a vector space. $\Rightarrow \sum_{i=1}^{m} U_i$ is the smallest subspace of $V$ containing all $U_j$'s. $\square$

## Direct sums

If $U_1, \dots, U_m \leq V$, then by definition we can write every $u \in \sum_{i=1}^m U_i$ as $u = u_1 + \dots + u_m$ with $u_i \in U_i$ for all $i$.

**Def 1.40** Direct sums

Let $U_1, \dots, U_m \leq V$ be subspaces of a vector space $V$. The sum $U_1 + \dots + U_m$ is called a direct sum, denoted $U_1 \oplus \dots \oplus U_m$, if every element $u \in \sum_{i=1}^m U_i$ can be written in a <u>unique</u> way as $u = u_1 + \dots + u_m$, where $u_i \in U_i$ for $1 \leq i \leq m$.

Examples:

1) $V = \mathbb{F}^3$. $U = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x, y \in \mathbb{F} \right\}$, $W = \left\{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} : z \in \mathbb{F} \right\}$

then $\mathbb{F}^3 = U \oplus W$.

2) $U$ and $W$ as above, and $X = \left\{ \begin{pmatrix} 0 \\ y \\ y \end{pmatrix} : y \in \mathbb{F} \right\}$ :

Then $\mathbb{F}^3 = U + W + X$, but <u>not</u> as a direct sum:

$$\mathbb{F}^3 \ni \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\in U} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\in W} + \underbrace{\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}}_{\in X}$$

**Prop 1.44** Suppose $U_1, \ldots, U_m \leq V$

Then $U_1 + \ldots + U_m$ is a direct sum

$\iff$ if $0 = u_1 + \ldots + u_m$ with $u_i \in U_i$, then $u_i = 0$ for all $i$.

Proof: $(\Rightarrow)$ clear, since $0 = 0 + \ldots + 0$ ($0 \in U_i$ for all $i$)

$(\Leftarrow)$ Let $u \in \sum_{i=1}^{m} U_i$, $u = u_1 + \ldots + u_m = v_1 + \ldots + v_m$, $u_i, v_i \in U_i$

$$0 = \underbrace{(u_1 - v_1)}_{\in U_1} + \underbrace{(u_2 - v_2)}_{\in U_2} + \ldots + \underbrace{(u_m - v_m)}_{\in U_m}$$

$\Rightarrow$ $u_i - v_i = 0$ for all $i$ by assumption $\Rightarrow$ $u_i = v_i$ for all $i$

$\Rightarrow$ $\sum_{i=1}^{m} U_i$ is a direct sum. $\square$

**Prop 1.45** Let $U, W \leq V$. Then $U + W = U \oplus W$ if and only if $U \cap W = \{0\}$.

Proof: $(\Rightarrow)$ Let $v \in U \cap W$ $\Rightarrow$ $v \in U$, $-v = (-1)v \in W$

But now $0 = \underbrace{v}_{\in U} + \underbrace{(-v)}_{\in W}$ $\Rightarrow$ $v = 0$ by Prop 1.44.

$\Rightarrow$ $U \cap W = \{0\}$.

$(\Leftarrow)$ Use Prop 1.44: assume $0 = u + w$ for some $u \in U$, $w \in W$.

$\Rightarrow$ $u = -w \in W$ and hence $u \in U \cap W = \{0\}$

$\Rightarrow$ $u = 0 = w$ $\Rightarrow$ $U + W = U \oplus W$ by Prop 1.44. $\square$

Caution: Prop 1.45 does <u>not</u> generalize to three or more summands!

See Ex. 2 above: $U \cap W = W \cap X = U \cap X = \{0\}$,

but $\mathbb{F}^3 = U + W + X$ is <u>not</u> a direct sum.