Lecture 5: Vector spaces

Last time: Fields, definition, examples

So far: vector spaces over $\mathbb{R}$

$\boxed{\text{Def 1.19}}$ Vector spaces

A vector space $(V,+,\cdot)$ over a field $\mathbb{F}$ is a set V together with a vector addition $+: V\times V \to V$ and a scalar multiplication $\cdot : \mathbb{F}\times V \to V$ satisfying

(V1) Associativity of $+$: for all $u,v,w\in V$, $(u+v)+w = u+(v+w)$

(V2) Neutral element for $+$: there exists $0\in V$ $v+0=v$ for all $v\in V$.

(V3) Inverse for $+$: for all $v\in V$ there exists a $w\in V$ $v+w=0$.

(V4) Commutativity of $+$: for all $v,w\in V$, $v+w=w+v$.

(S1) Associativity of $\cdot$: for $a,b\in\mathbb{F}$ and $v\in V$, $a\cdot(b\cdot v) = (ab)\cdot v$

(S2) Neutral element for $\cdot$: the scalar $1\in\mathbb{F}$ satisfies $1\cdot v = v$ for all $v\in V$.

(D1) Distributivity 1: for all $a\in\mathbb{F}$, $v,w\in V$, $a\cdot(v+w) = a\cdot v + a\cdot w$

(D2) Distributivity 2: for all $a,b\in\mathbb{F}$, $v\in V$, $(a+b)\cdot v = a\cdot v + b\cdot v$

Remarks: ·) (V1) - (V4) say that $(V,+)$ is an (additive) Abelian group.

·) We will often write $av = a\cdot v$ for $a\in\mathbb{F}$, $v\in V$.

Examples:

- $\mathbb{F}$ field, $n \in \mathbb{N}$: $\mathbb{F}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} ; x_i \in \mathbb{F} \text{ for } 1 \leq i \leq n \right\}$

  e.g., $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{F}_2^n$, ...

  We will prove later that every finite-dim. vector space over a field $\mathbb{F}$ is "isomorphic" to $\mathbb{F}^n$ for some $n \in \mathbb{N}$.

  vector addition: $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{F}^n$

$$x + y = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

  scalar multiplication: $a \in \mathbb{F}$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n$: $ax = \begin{pmatrix} a x_1 \\ \vdots \\ a x_n \end{pmatrix}$

  $(d \in \mathbb{N})$

- polynomials of degree at most d with coefficients in $\mathbb{F}$:

$$P_d(\mathbb{F}) = \left\{ a_0 + a_1 X + \ldots + a_d X^d : a_i \in \mathbb{F},\ i = 0, \ldots, d \right\}$$

  Ex.: $p_1 = 2 + 3x^2 + x^3$, $p_2 = 4x + 7x^2 + 2x^3$

  $p_1 + p_2 = 2 + 4x + 10x^2 + 3x^3$ (addition)

  $3p_1 = 6 + 9x^2 + 3x^3$ (scalar multiplication)

We can identify a polynomial $p = a_0 + a_1 x + \ldots + a_d x^d$

with $\tilde{p} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_d \end{pmatrix} \Rightarrow P_d(\mathbb{F}) \cong \mathbb{F}^{d+1}$

·) $C([a,b]) = \{ f : [a,b] \to \mathbb{R} : f \text{ continuous} \}$, $a, b \in \mathbb{R}$, $a < b$

continuous real-valued functions on $[a,b]$

vector addition and scalar multiplication are defined point-wise:

$f, g \in C([a,b])$: $\quad f+g : [a,b] \to \mathbb{R}$
$$x \mapsto f(x) + g(x)$$

$\lambda \in \mathbb{R}$: $\quad \lambda \cdot f : [a,b] \to \mathbb{R}$
$$x \mapsto \lambda f(x)$$

Easy exercise: $f+g, \lambda f \in C([a,b])$ for $f, g \in C([a,b])$, $\lambda \in \mathbb{R}$

·) $m \times n$ matrices with entries in a field $\mathbb{F}$:

$$M_{m,n}(\mathbb{F}) = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} : a_{ij} \in \mathbb{F} \text{ for } 1 \le i \le m, 1 \le j \le n \right\}$$

vector addition and scalar multiplication are defined entry-wise.

·) functions from a set S to a field $\mathbb{F}$

$$\mathbb{F}^S = \{ f : S \to \mathbb{F} \}$$

vector addition and scalar multiplication are defined point-wise:

$f, g \in \mathbb{F}^S$: $(f+g)(s) := f(s) + g(s)$, $\lambda \in \mathbb{F}$: $(\lambda f)(s) := \lambda f(s)$

For $S = \mathbb{N}$, elements of $\mathbb{F}^{\mathbb{N}}$ are called <u>sequences</u>.

$\boxed{\text{Prop}}$ (1.25, 1.26, 1.29, 1.30)

Let V be a vector space over a field $\mathbb{F}$.

i) The additive neutral element in V is unique.

ii) Every element $v \in V$ has a unique additive inverse, denoted by $-v$.

iii) $0v = 0$ for all $v \in V$ (recall that $\mathbb{F} = (\mathbb{F}, +, \cdot, 0, 1)$)

iv) $a0 = 0$ for all $a \in \mathbb{F}$ (where 0 is the unique 0-elem. from i)).

v) Let -1 denote the additive inverse of $1 \in \mathbb{F}$.

Then $(-1)v = -v$ for all $v \in V$.

<u>Proof:</u> i) Let $0, 0'$ be two additive neutral elements in V:

$$0 = 0 + 0' = 0'.$$

ii) Let w, w' be two inverses of $v \in V$:

$$w = w + 0 = w + (v + w') = \underbrace{(w+v)}_{0} + w' = w'.$$

iii) $v \in V$: $0v = (0+0)v = 0v + 0v \Rightarrow 0v = 0$.

iv) $a \in \mathbb{F}$: $a0 = a(0+0) = a0 + a0 \Rightarrow a0 = 0$.

v) $v \in V$: $$v + \underset{\substack{\uparrow \\ \text{add. inverse} \\ \text{of } 1 \in \mathbb{F}}}{(-1)}v = 1v + (-1)v = \underbrace{(1 + (-1))}_{= 0 \in \mathbb{F}} v = 0v \overset{\text{(iii)}}{=} 0$$

$$\Rightarrow (-1)v = -v \quad \forall v \in V. \qquad \square$$