Lecture 2: Gaussian elimination

Last time: examples of vector spaces, linear maps, linear equations

System of linear equations over $\mathbb{R}$: n variables, m equations

$$\begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n &= b_m \end{aligned} \qquad a_{ij}, b_i \in \mathbb{R} \text{ for } 1 \le i \le m,\ 1 \le j \le n$$

Goal: Find solution(s) $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ s.t. $\sum_{j=1}^{n} a_{ij} c_j = b_i$ for $1 \le i \le m$.

In order to solve a sys. of lin. eq., we can:

a) change the order of equations

b) multiply both sides of an eq. by (non-zero) scalar

c) add one eq. to another one

These operations do not change the set of solutions (HW 1)

<u>Gaussian elimination</u>: algorithmic way of finding solutions of sys. of lin. eq.'s using a), b), c)

Variables x_j are just placeholders, and we work with the following

<u>augmented matrix of coefficients:</u> $\quad m \times (n+1)$ matrix

$$\begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &= b_1 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n &= b_m \end{aligned} \longrightarrow \text{m rows} \hat{=} \text{m eq's} \left\{ \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right) \right.$$

$\underbrace{\qquad\qquad\qquad}$ n columns $\hat{=}$ n variables

The allowed operations a), b), c) translate to:

a') change order of the rows of the matrix

b') multiply a row of the matrix by (non-zero) scalar

c') add one row of the matrix to another one.

Goal of Gaussian elimination: Use a'), b'), c') to bring the augmented matrix of coefficients into the following form:

$$\left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array} \right) \xrightarrow{a'),\, b'),\, c')} \left(\begin{array}{cccc|c} 1 & 0 & & & c_1 \\ 0 & 1 & & & c_2 \\ \vdots & 0 & & \dots & \vdots \\ & \vdots & & & \\ 0 & 0 & & & c_m \end{array} \right) \qquad \begin{array}{l} x_1 = c_1 \\ x_2 = c_2 \\ \vdots \end{array}$$

Under the right circumstances, $(c_1, \dots, c_m)^T$ is a solution!

Ex.: 3 variables, 3 eq's.

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 39 \\ 2x_1 + 3x_2 + x_3 &= 34 \\ x_1 + 2x_2 + 3x_3 &= 26 \end{aligned} \longrightarrow \left(\begin{array}{ccc|c} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 26 \\ 2 & 3 & 1 & 34 \\ 3 & 2 & 1 & 39 \end{array}\right) \xrightarrow[R_3 \leftarrow R_3 - 3R_1]{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 26 \\ 0 & -1 & -5 & -18 \\ 0 & -4 & -8 & -39 \end{array}\right)$$

$$\xrightarrow{R_2 \leftarrow -R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 26 \\ 0 & 1 & 5 & 18 \\ 0 & -4 & -8 & -39 \end{array}\right) \xrightarrow[R_3 \leftarrow R_3 + 4R_2]{R_1 \leftarrow R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & -7 & -10 \\ 0 & 1 & 5 & 18 \\ 0 & 0 & 12 & 33 \end{array}\right)$$

$$\xrightarrow{R_3 \leftarrow R_3/12} \left(\begin{array}{ccc|c} 1 & 0 & -7 & -10 \\ 0 & 1 & 5 & 18 \\ 0 & 0 & 1 & 11/4 \end{array}\right) \xrightarrow[R_2 \leftarrow R_2 - 5R_3]{R_1 \leftarrow R_1 + 7R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 37/4 \\ 0 & 1 & 0 & 17/4 \\ 0 & 0 & 1 & 11/4 \end{array}\right)$$

We have found a solution $c_1 = 37/4$, $c_2 = 17/4$, $c_3 = 11/4$!

We call this form of a matrix a row-echelon form (REF):

1) Every row containing a non-zero entry is above every row with all zeros

2) The first non-zero entry of every row is 1.

3) If $i<j$, then the first non-zero entry of row i is strictly to the left of the first non-zero entry of row j.

(first non-zero element in a row is called a <u>pivot</u>)

Furthermore, an REF matrix is in reduced row-echelon form if (RREF)

4) Any column containing a pivot contains no other non-zero elements.

<u>Examples:</u>

$$\begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 4 \end{pmatrix}$$ REF, but not RREF

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$ neither

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$ RREF

$\boxed{\text{Thm}}$ (Gaussian elimination)

Every matrix can be brought into reduced row-echelon form.

Proof idea: Upgrade previous example to an algorithmic proof that always returns a matrix in RREF. "□"

Solutions of sys. of lin. eq. do not always exist!

$$\begin{aligned} 3x_1 + 4x_2 &= 6 \\ 3x_1 + 4x_2 &= 7 \end{aligned} \longrightarrow \left(\begin{array}{cc|c} 3 & 4 & 6 \\ 3 & 4 & 7 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 4/3 & 2 \\ 0 & 0 & 1 \end{array}\right)$$

$$\Rightarrow \quad x_1 + \tfrac{4}{3} x_2 = 2$$

$$0 = 1 \ \lightning.$$

This system is inconsistent (no solutions)

$\Leftrightarrow$ RREF has a pivot in the last column.