WELCOME TO MATH 416: ABSTRACT LINEAR ALGEBRA!

Lecture 1: Introduction

What is linear algebra? It's the study of:

a) vector spaces: algebraic structure consisting of objects called vectors that we can add together and multiply by a scalar.

b) linear maps: maps between vector spaces that preserve the linear structure of vector spaces.

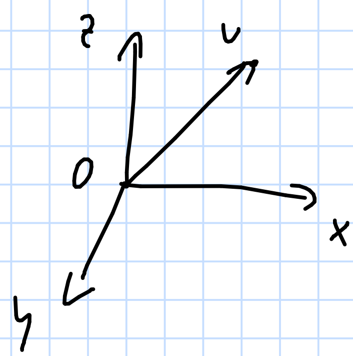
Important example of a vector space (a):

Euclidean space $\mathbb{R}^n$

e.g. $n=1$: real line

$n=2$: real plane

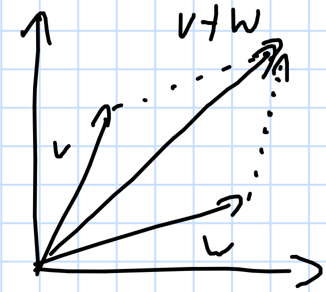
$n=3$: real 3D space



In $\mathbb{R}^n$, vectors are identified with "arrows" pointing from the origin 0 to a point with coordinates $(x_1, \dots, x_n)$ $(x_i \in \mathbb{R})$:

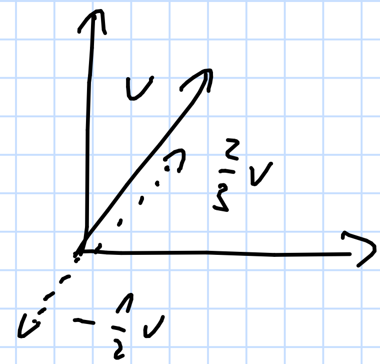
$$v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

·) we can add vectors together:



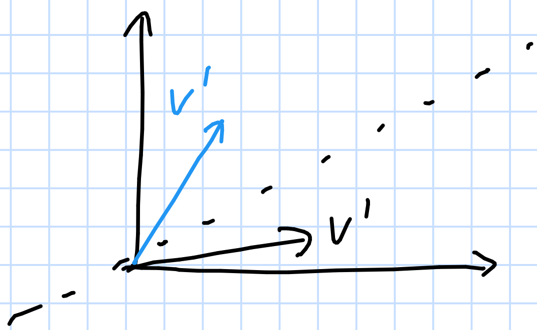
·) we can multiply vectors by a scalar $\lambda \in \mathbb{R}$

→ stretching / shortening / flipping



What are some examples of linear maps (L)?

·) Reflection across a line in $\mathbb{R}^2$:

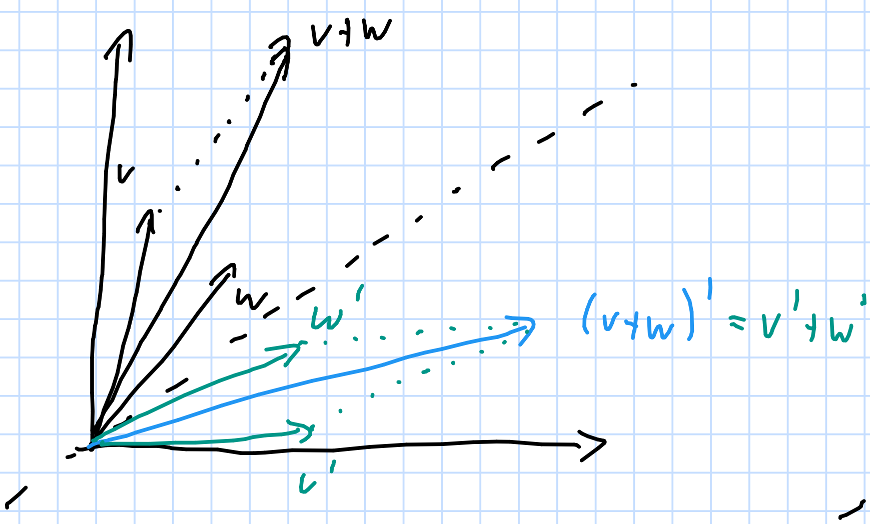


Linearity of a map:

applying the map commutes

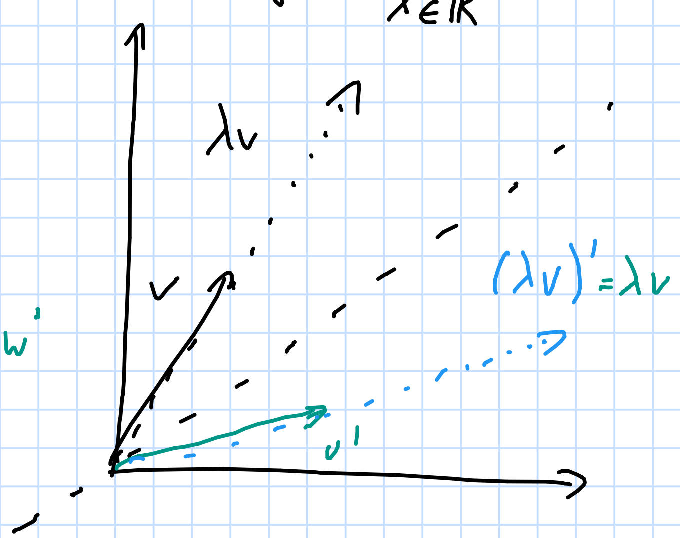
with addition and multiplication by a scalar.

addition:

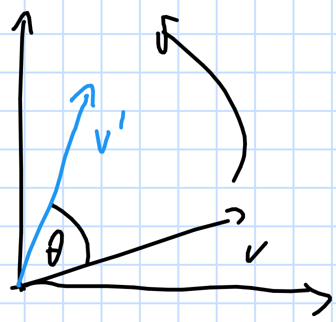


scaling:

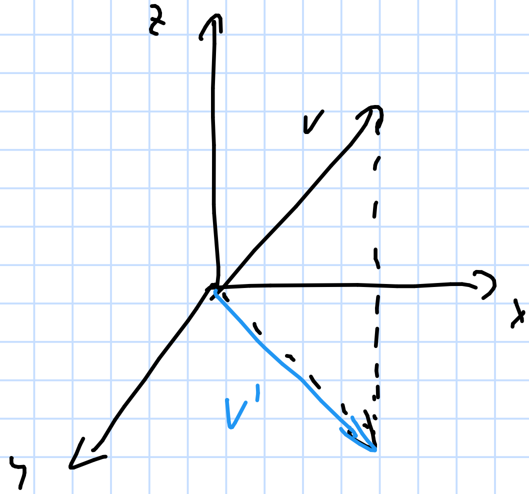
$\lambda \in \mathbb{R}$



·) Rotation around the origin by a fixed angle θ:



·) Projections: Ex: project vectors in $\mathbb{R}^3$ onto the xy-plane



$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto v' = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

·) more abstract example: differentiation of polynomials

$$p(x) = x^4 + 3x^2 + 7x \longmapsto p'(x) = 4x^3 + 6x + 7$$

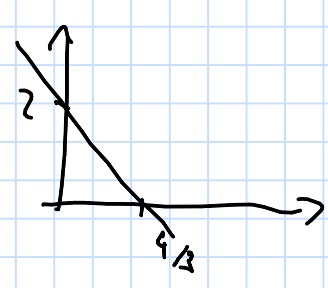
Linearity: $(p+q)' = p' + q'$, $(\lambda p)' = \lambda p'$.

Linear equations

Remember: a linear equation defines an (affine) linear subspace of $\mathbb{R}^n$.

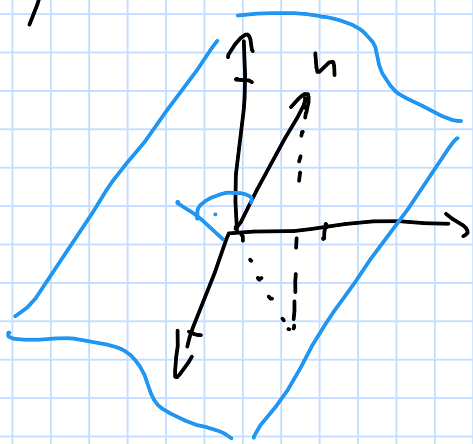
E.g., $3x + 2y = 4$ describes a line in $\mathbb{R}^2$

$$y = -\frac{3}{2}x + 2$$



·) in $\mathbb{R}^3$, a plane can be defined by its normal vector $n = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$

E.g., $n = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \longleftrightarrow E: x + 2y + 3z = 0$



·) more generally, in $\mathbb{R}^n$ an eq. of the form $a_1 x_1 + \dots + a_n x_n = 0$ with $a_i \in \mathbb{R}$ describes a hyperplane of dim. $n-1$ (co-dim. 1)

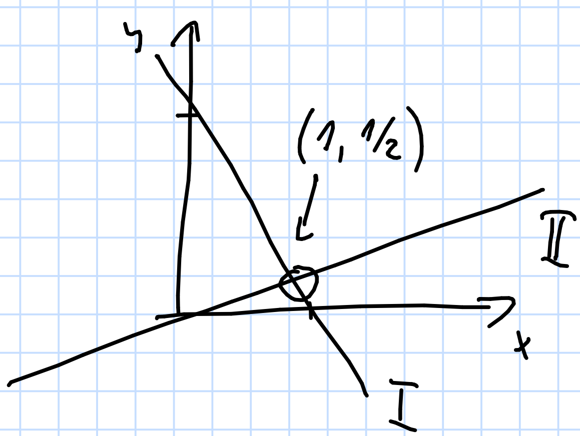
Given two or more of these hyperplanes, we can ask if / how they intersect $\longleftrightarrow$ is there a solution to this system of linear equations?

Ex.: $\mathrm{I}: 3x + 2y = 4$

$\mathrm{II}: x - 2y = 0$

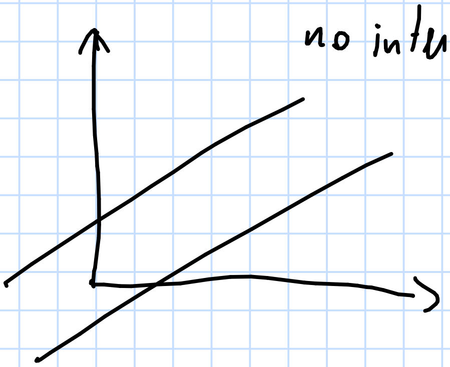
$4x = 4$

$\Rightarrow x = 1, \; y = \frac{1}{2}$



Sometimes there is no intersection / solution, or infinitely many.

in $\mathbb{R}^2$:



We will tackle the following general question:

Given m equations in n variables (geom.: m hyperplanes in $\mathbb{R}^n$)

$$
\begin{aligned}
a_{11} x_1 + \dots + a_{1n} x_n &= b_1 \\
&\vdots \\
a_{m1} x_1 + \dots + a_{mn} x_n &= b_m ,
\end{aligned}
\qquad a_{ij}, b_i \in \mathbb{R} \text{ for } \begin{array}{l} 1 \le i \le m \\ 1 \le j \le n \end{array}
$$

is there a solution, i.e., $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ s.t. $\sum_{j=1}^{n} a_{ij} c_j = b_i$ for $1 \le i \le m$?