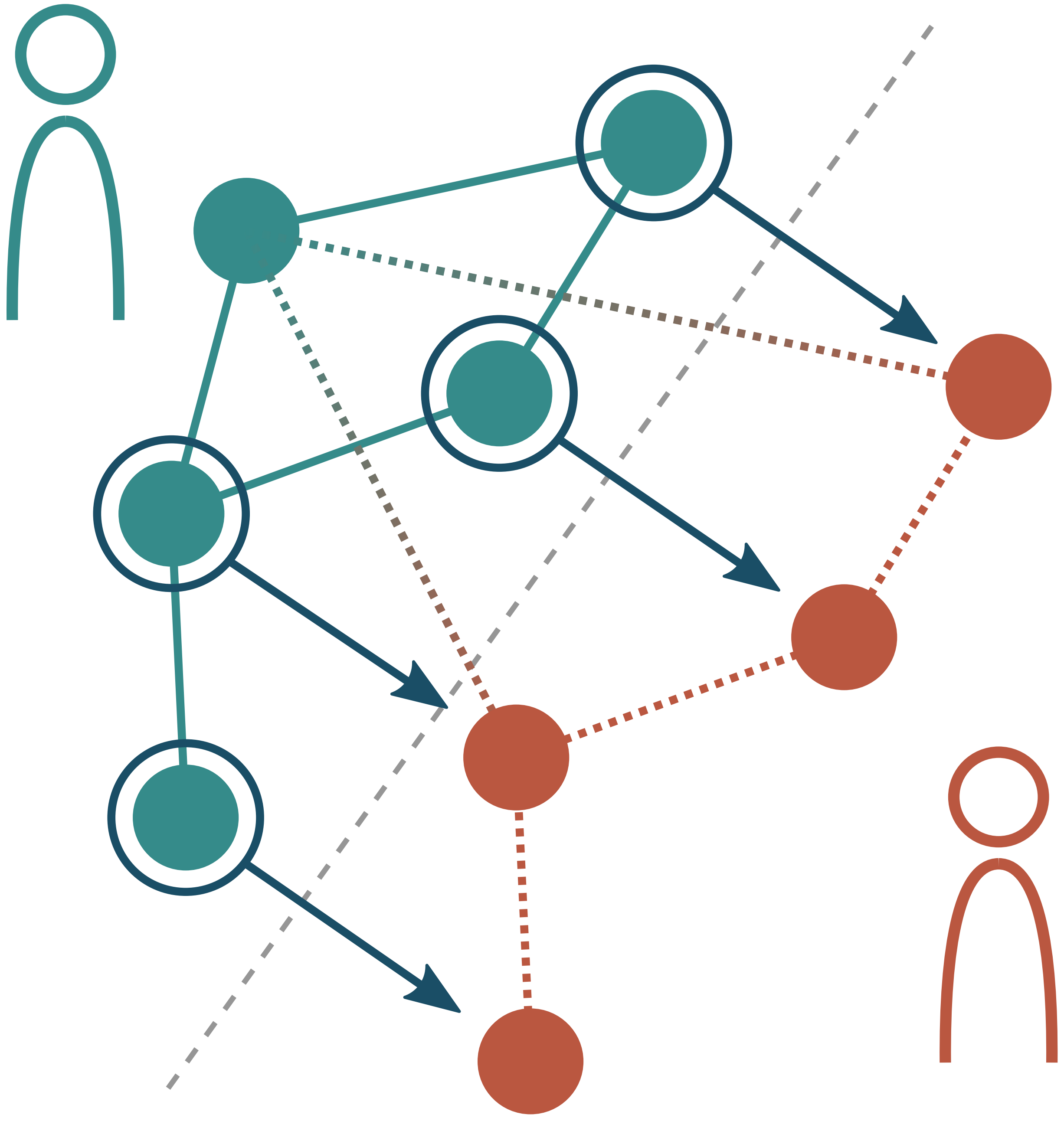
Beyond IID 8 - extended talk November 6, 2020

Error thresholds for arbitrary Pauli noise

arXiv:1910.00471



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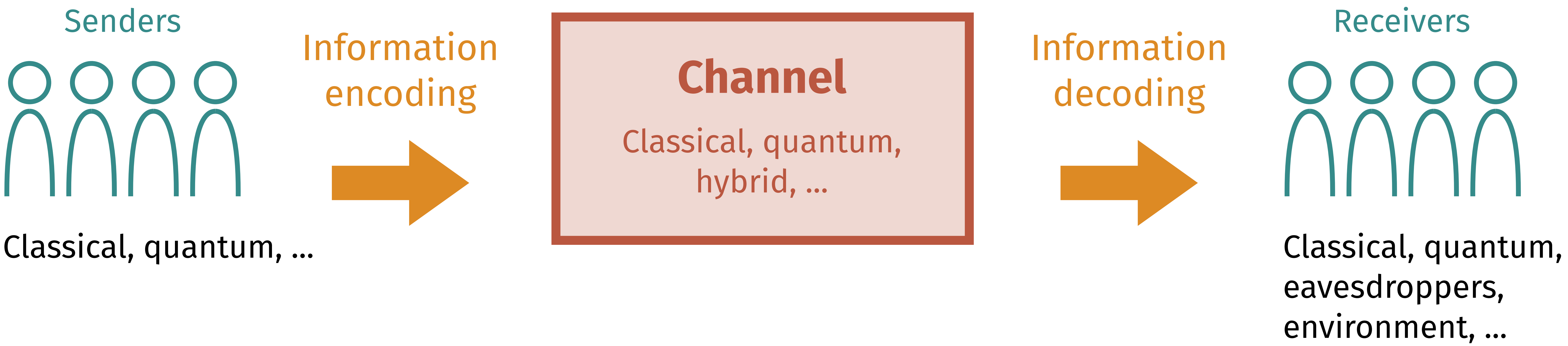
(joint work with J. Bausch, University of Cambridge)

Channel coding in information theory

Many communication tasks can be formalized as a channel coding problem:



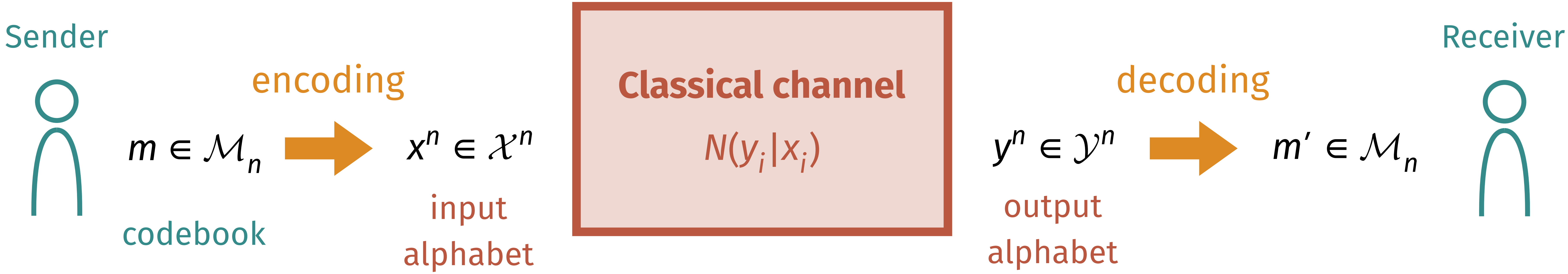
Central notions in information theory:

→ **Capacity of a channel:** quantifies information-processing capabilities of a channel.

→ **Coding theorem:** expresses capacity as (optimization over) entropic quantities.

Channel coding in information theory

Prototypical (and first) example: classical point-to-point channel



Capacity:

$$\mathcal{C}(N) := \sup\{ \tfrac{1}{n} \log |\mathcal{M}_n| : \ \Pr(M \neq M') \to 0\}.$$

Shannon entropy: $H(X) = -\sum_{x \in \mathcal{X}} p_X(x) \log p_X(x)$

Mutual information: $\mathcal{I}(X;Y) = H(X) + H(Y) - H(XY)$

Shannon's noisy channel coding theorem

$$\mathcal{C}(N) = \max_{p_X} \mathcal{I}(X;Y)$$

[Shannon '48]

Channel coding in information theory

Point-to-point classical communication is **extremely well understood**:

Shannon's formula is **single-letter**, i.e., a bounded optimization problem.

$$\mathcal{C}(N) = \max_{p_X} \mathcal{I}(X;Y)$$

[Shannon '48]

Shannon's theorem can be phrased as a **geometric program**, a type of convex program.

[Chiang, Boyd '04]

Capacity of a classical channel can be **efficiently computed** in time

$$O(|\mathcal{Y}||\mathcal{X}| \log |\mathcal{X}| \varepsilon^{-1}).$$

[Arimoto '72; Blahut '72]

There are families of **capacity-achieving codes** with efficient encoding/decoding: LDPC codes, turbo codes, polar codes.

[Gallager '60; Berrou '91; Arıkan '09]

Channel coding in information theory

Problematic settings

- → Network information theory (anything beyond 1 sender → 1 receiver)
- → Quantum resources: quantum channels, quantum information, …

Complications

- → Increased complexity of algorithms
- → Non-convex optimization problems
- → Unbounded optimization problems (multi-letter formulas)

This talk: Quantum information transmission through quantum channel

- → Relevant capacity: **Quantum capacity** of a quantum channel
- → **Non-convexity** and **multipartite entanglement** main problems/objects of study.
- → Use mathematical/numerical tools, in particular **symmetries** and **optimization techniques,** to study quantum capacity.

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Quantum capacity of a quantum channel

Quantum channel models:

Information theory

Noisy communication link between quantum parties.

Error correction

Environmental noise in a quantum device.

Quantum capacity characterizes:

How much quantum information can be sent faithfully?

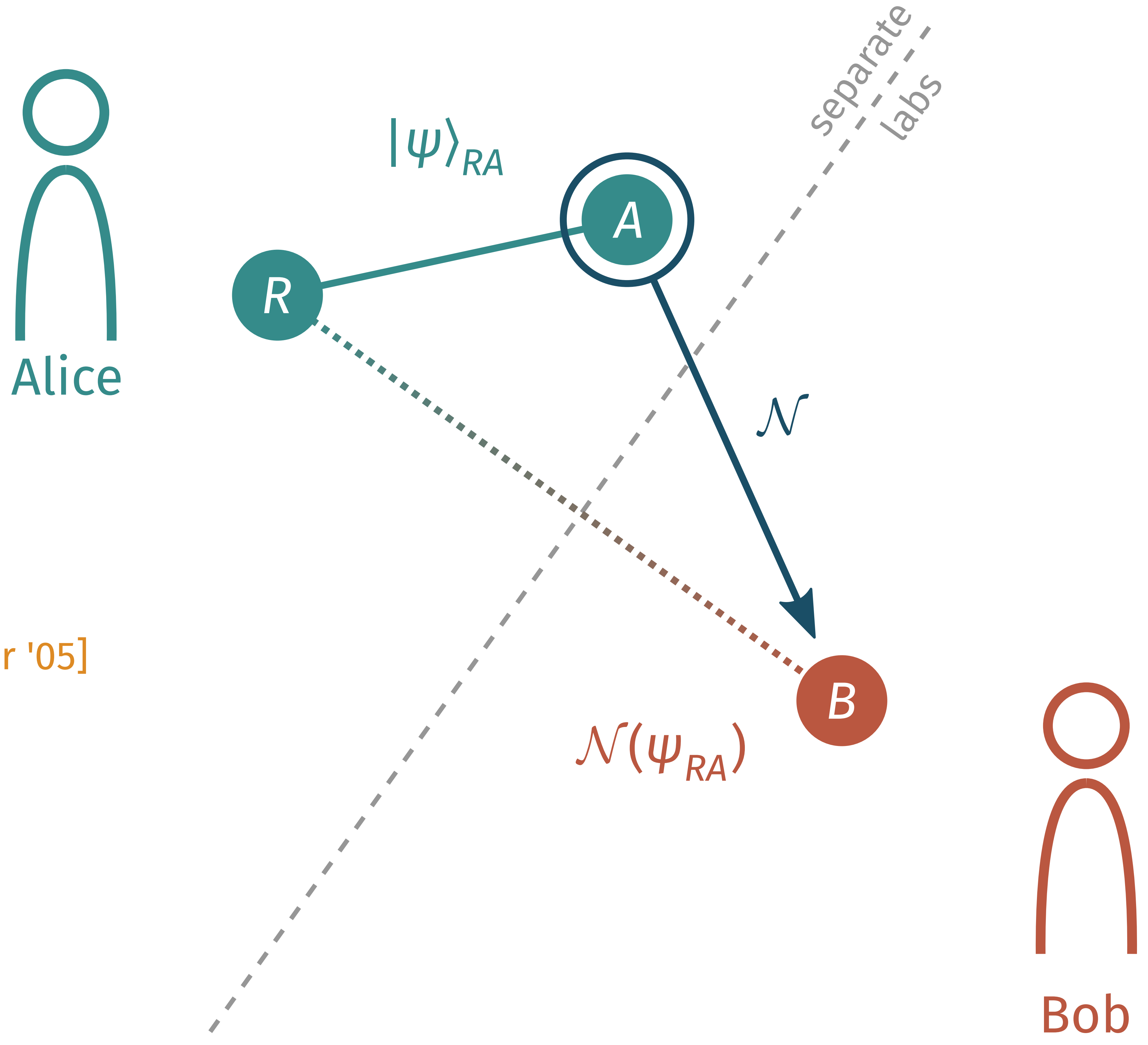
How much quantum information can be protected against noise?

Quantum capacity of a quantum channel

Entanglement generation:

→ Share k identical copies of pure bipartite state ψ_{RA} via channel $\mathcal{N}$.

→ Distill EPR pairs $|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B$ from $\mathcal{N}(\psi_{RA})^{\otimes k}$ using **local operations** and **forward classical communication** with vanishing error. [Devetak '05; Devetak, Winter '05]



Achievable rate: coherent information

$$\mathcal{I}(\psi,\mathcal{N}) = S(\mathcal{N}(\psi_A)) - S(\mathcal{N}(\psi_{RA}))$$

Von Neumann entropy: $S(\rho) = -\operatorname{tr}(\rho \log \rho)$

Quantum capacity of a quantum channel

Idea:

→ Distribute a multipartite state ψ_{RA^n} via n identical and independent copies of $\mathcal{N}$.

→ Rate for distilling from $[\mathcal{N}^{\otimes n}(\psi_{RA^n})]^{\otimes k}$:

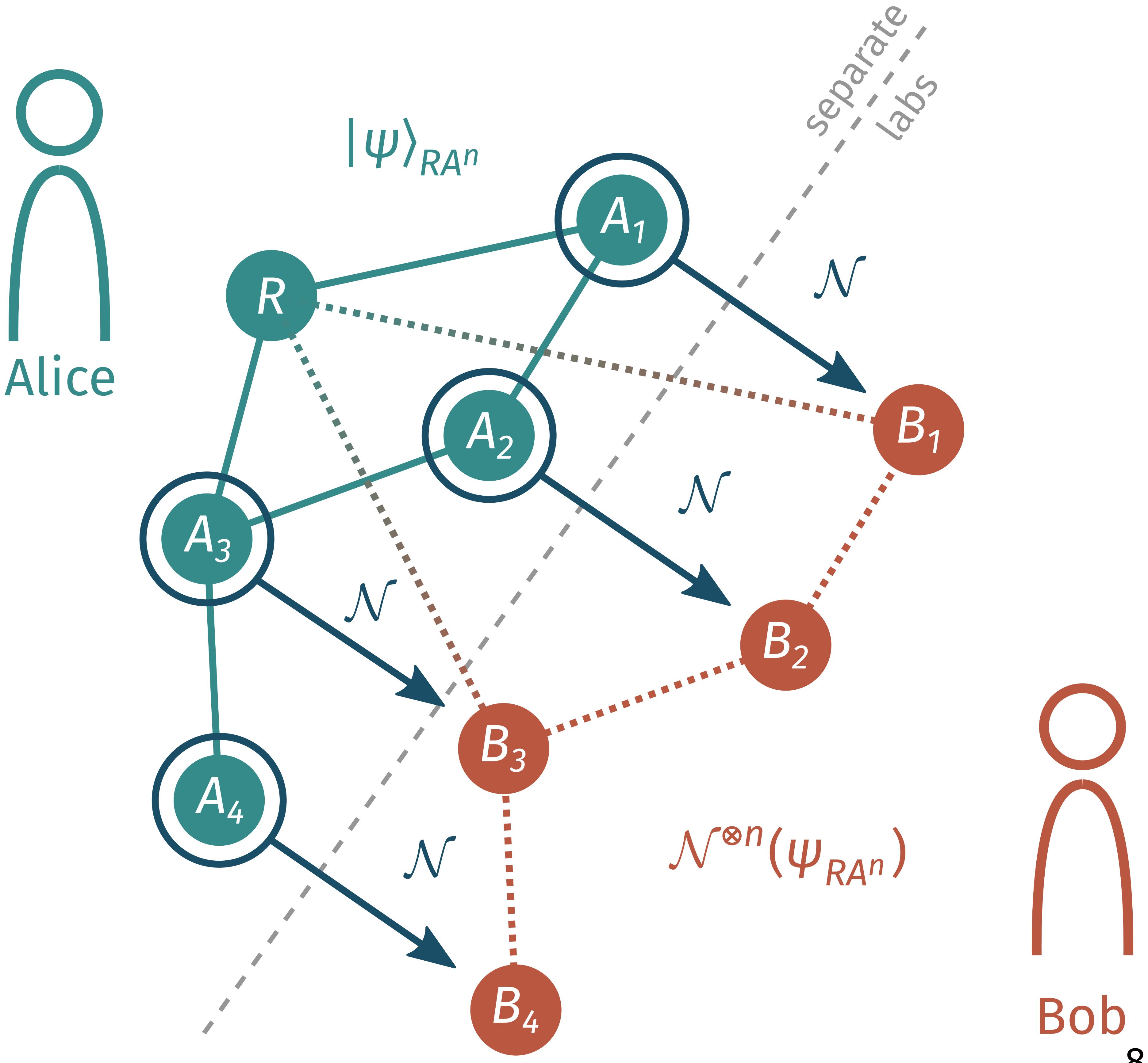
$$\frac{1}{n}\mathcal{I}_c(\psi_n, \mathcal{N}^{\otimes n})$$

Superadditivity of coherent information

There are ψ_n and $\mathcal{N}$ such that

$$\frac{1}{n}\mathcal{I}_c(\psi_n, \mathcal{N}^{\otimes n}) > \max_{\phi} \mathcal{I}_c(\phi, \mathcal{N}).$$

[Shor, Smolin '96; DiVincenzo et al. '98]



Quantum capacity of a quantum channel

Quantum capacity $Q(\mathcal{N})$**:** largest rate at which EPR pairs can be generated with asymptotically vanishing error.

Quantum capacity coding theorem

$$Q(\mathcal{N}) = \sup_{n\in\mathbb{N}} \frac{1}{n} \max_{\psi_n} \mathcal{I}_c(\psi_n, \mathcal{N}^{\otimes n})$$

Unbounded optimization problem (because of superadditivity) and known pathological behavior.

Non-concave maximization problem for channels with superadditive coherent information.

[Schumacher '96; Schumacher, Nielsen '96; Lloyd '97; Shor '02; Devetak '05; Watanabe '12; Cubitt '15]

Quantum capacity of a quantum channel

Quantum capacity

$$Q(\mathcal{N}) = \sup_{n\in\mathbb{N}} \frac{1}{n} \max_{\psi_n} \mathcal{I}_c(\psi_n, \mathcal{N}^{\otimes n})$$

→ **Multipartite entanglement** in code state ψ_{RA^n} causes superadditivity.

→ Characterizing multipartite entanglement is hard for growing n due to **exponential scaling** of Hilbert space dimension.

→ For fixed n, objective function is non-concave and hard to optimize.

Possible computational ansatz

Restrict quantum states to **polynomial subspace** of Hilbert space with sufficiently rich entanglement structure, such as quantum neural network states.

[Bausch, FL '18]

Possible mathematical ansatz

For specific quantum channels, consider **symmetric codes** and exploit symmetries to compute coherent information.

➙ Permutation invariance [Kern, Renes '08]

➙ **Graph symmetries** [Bausch, FL '19]

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Pauli channels and stabilizer states

Pauli channels

$\mathcal{N}_{\mathrm{p}}(\rho) = p_0\rho + p_1 X\rho X + p_2 Y\rho Y + p_3 Z\rho Z$

→ $\mathbf{p} = (p_0, p_1, p_2, p_3)$ probability distribution.

→ X, Y, Z Pauli matrices.

Important examples:

depolarizing noise $\mathbf{p}_{\mathrm{dep}} = (1-p, \frac{p}{3}, \frac{p}{3}, \frac{p}{3})$

BB84 channel $\mathbf{p}_{\mathrm{BB84}} = ((1-p)^2, p-p^2, p^2, p-p^2)$

→ QEC: Ability to correct X, Y, Z errors is sufficient for correcting **arbitrary unitary errors**. [Shor '95; Steane '96]

→ For Pauli channels of the form $\mathrm{p}_x = (1-x, xp_1, xp_2, xp_3)$ we are interested in the **threshold**:

$$\text{Supremum over all } x \text{ such that } Q(\mathcal{N}_{\mathrm{p}_x}) > 0.$$

QIT: $Q(\mathcal{N}) > 0 \Leftrightarrow$
faithful quantum communication possible.

QEC: $Q(\mathcal{N}) > 0 \Leftrightarrow$
perfect error-correcting code exists.

Pauli channels and stabilizer states

→ Pauli channel: $\mathcal{N}_{\mathbf{p}}(\rho) = p_0\rho + p_1 X\rho X + p_2 Y\rho Y + p_3 Z\rho Z$.

→ Tensor powers: $\mathcal{N}_{\mathrm{p}}^{\otimes n}(\rho_n) = \sum_{j^n} p_{j^n} E_{j^n} \rho_n E_{j^n}$,

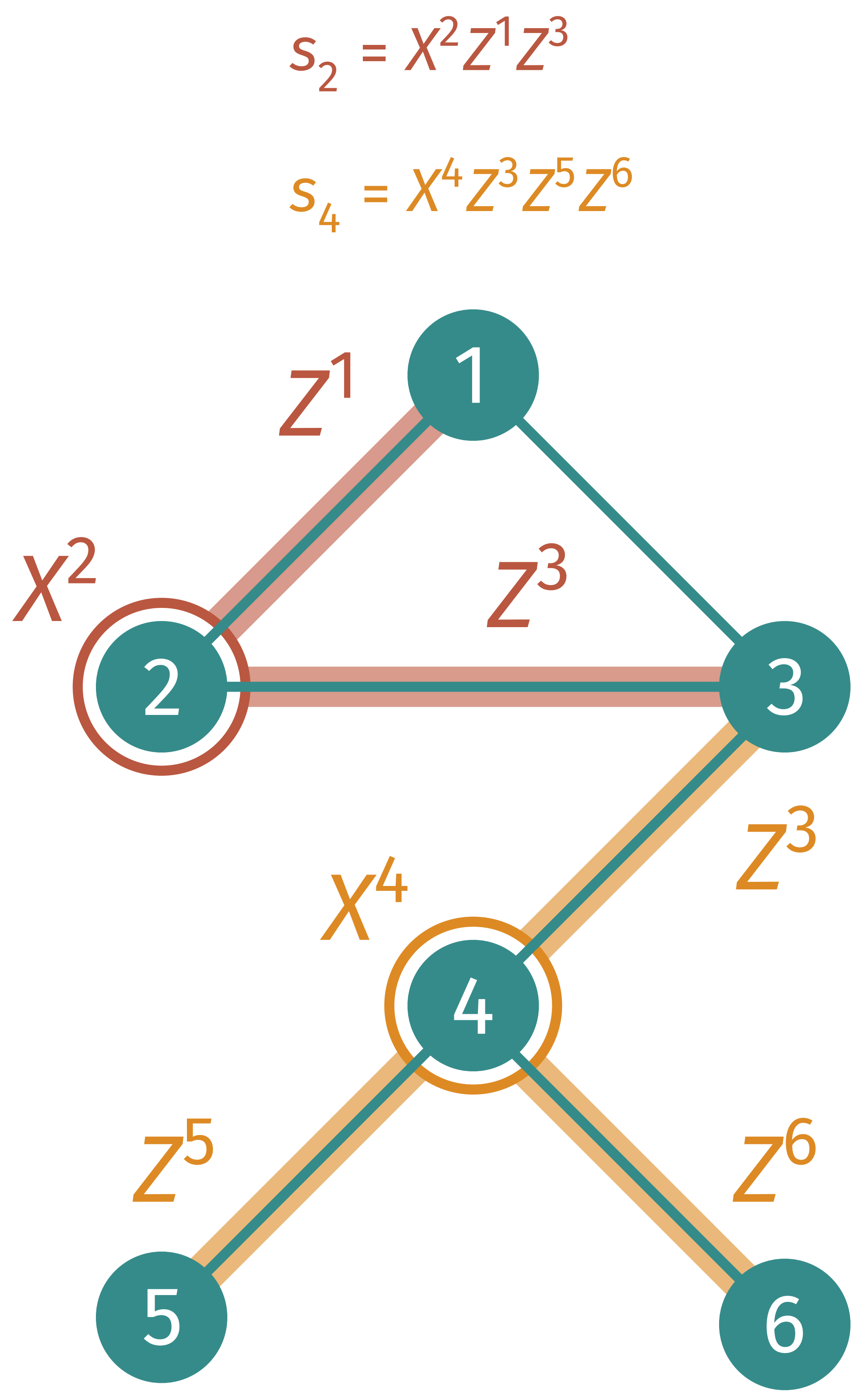
with n-qubit Pauli operators $E_{j^n} \in \mathcal{P}_n = \{I, X, Y, Z\}^{\otimes n} \cup \{\pm 1, \pm i\}$.

→ Restrict to **stabilizer states** $|\psi_k\rangle$ which are stabilized by

k pairwise commuting stabilizer generators $s_i \in \mathcal{P}_k$:

$$s_i|\psi_k\rangle = |\psi_k\rangle \quad \text{for } i = 1, \dots, k.$$

[Gottesman '97]



Graph states

Let $\Gamma = (V, E)$ be a graph and $N_i = \{j \in V : (i,j) \in E\}$.

For each vertex i define stabilizers $s_i = X^i \prod_{j \in N_i} Z^j$.

The **graph state** $|\Gamma\rangle$ is the unique pure state stabilized by $s_1, \dots, s_k$.

Graph states

Every stabilizer state is local unitary (LU) equivalent to a graph state: [van den Nest et al. '04]

For all stab's $|\psi_k\rangle$ there exist $\Gamma = (V, E)$ and unitaries $U_1, \dots, U_k$ s.t. $U_1 \otimes \dots \otimes U_k |\psi_k\rangle = |\Gamma\rangle$.

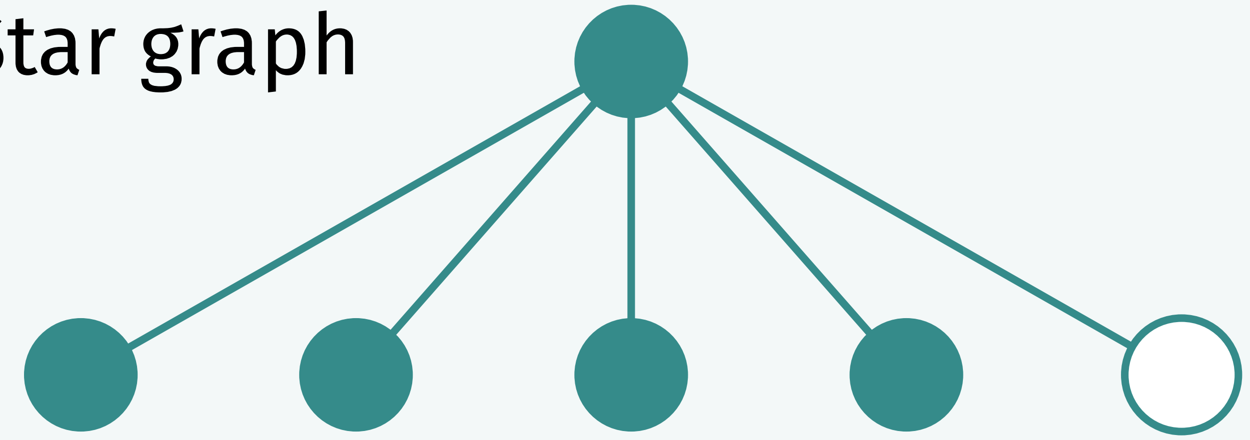
Important quantum codes as graph states:

● system qubits A_i ○ reference qubit R

Repetition code (GHZ state)

$|0\rangle_R \otimes |0\rangle_A^{\otimes n} + |1\rangle_R \otimes |1\rangle_A^{\otimes n}$

Star graph

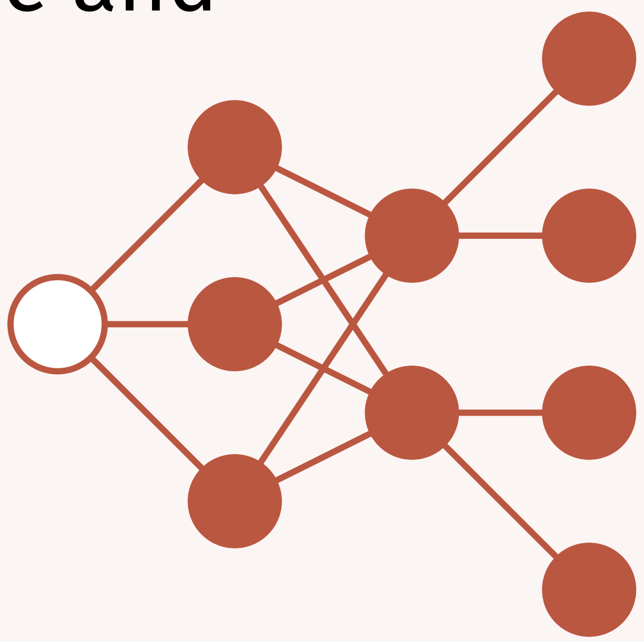


Cat code (Shor code)

Concatenation of Z-type and X-type rep code.

X-type rep: $|+\rangle^{\otimes n} + |-\rangle^{\otimes n}$,

where $|\pm\rangle \sim |0\rangle \pm |1\rangle$.



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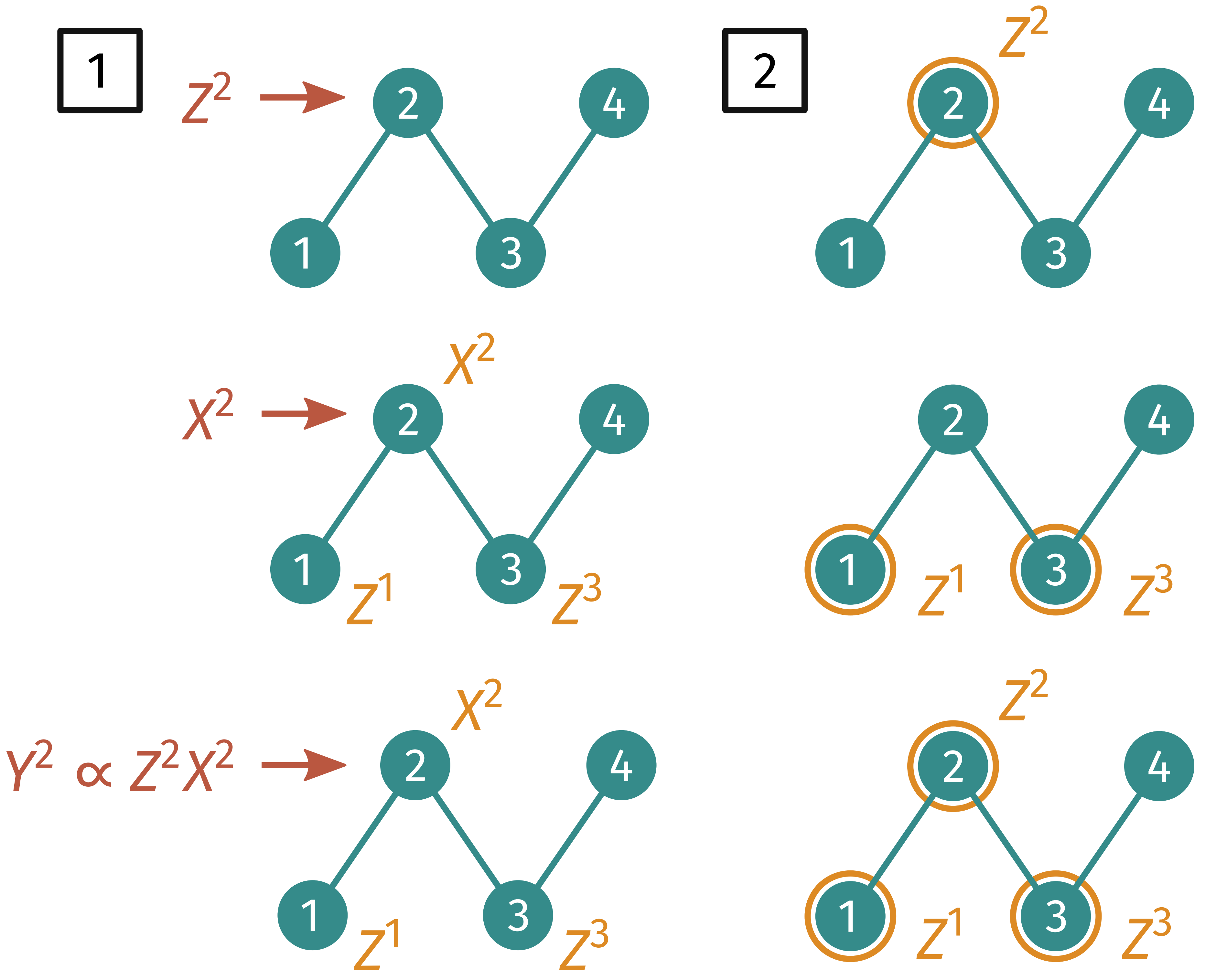
Decoherence of graph states

Quantum capacity: $Q(\mathcal{N}) = \sup_{n \in \mathbb{N}} \frac{1}{n} \max_{\psi_n} \mathcal{I}_c(\psi_n, \mathcal{N}^{\otimes n})$ $\Rightarrow$ $Q(\mathcal{N}) \geq \frac{1}{n} \mathcal{I}_c(\psi_n, \mathcal{N}^{\otimes n})$ for all ψ_n and $n \in \mathbb{N}$.

Coherent information: $\mathcal{I}_c(\phi, \mathcal{N}) = S(\mathcal{N}(\phi_A)) - S(\mathcal{N}(\phi_{RA}))$



Goal

Compute the tensor action of a Pauli channel

$$\mathcal{N}_{\mathbf{p}}(\rho) = p_0 \rho + p_1 X \rho X + p_2 Y \rho Y + p_3 Z \rho Z$$

on a graph state $|\Gamma\rangle$.

Observation

Graph states translate (products of) Pauli errors into Z-type errors.

Decoherence of graph states

Decoherence of graph states:

$$Z^i|\Gamma\rangle\langle\Gamma|Z^i = Z^i|\Gamma\rangle\langle\Gamma|Z^i \qquad X^i|\Gamma\rangle\langle\Gamma|X^i = Z^{N_i}|\Gamma\rangle\langle\Gamma|Z^{N_i} \qquad Y^i|\Gamma\rangle\langle\Gamma|Y^i = Z^{i\oplus N_i}|\Gamma\rangle\langle\Gamma|Z^{i\oplus N_i}$$

Graph states subjected to Pauli noise

For a Pauli channel, output state is of the form

$$\mathcal{N}^{\otimes n}(|\Gamma\rangle\langle\Gamma|) = \sum_{U\subset V} \lambda_U |U\rangle\langle U|,$$

where the $|U\rangle = Z^U|\Gamma\rangle$ form the **graph state basis**.

Computing coherent information

Determine coefficients λ_U

for all subset vertices $U \subset V$

using the decoherence rules above.

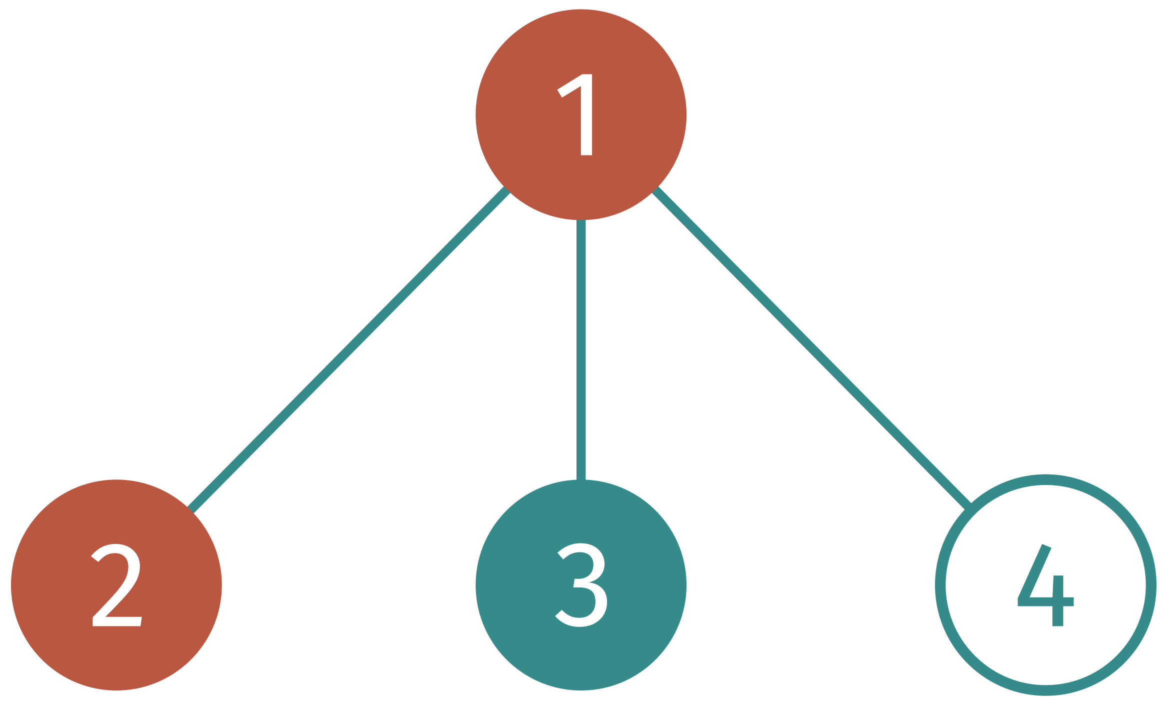
[Hein et al. '05]

Decoherence of graph states

Example: star graph on 4 vertices, $U = \{1,2\}$.

$$\mathcal{N}^{\otimes n}(|\Gamma\rangle\langle\Gamma|) = \sum_{U \subset V} \lambda_U |U\rangle\langle U|$$

Pauli error operators generating U:



$$\begin{array}{cccc} Z^1Z^2 & Y^2 & Z^2X^3 & Z^1Y^2X^3 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$

$$\lambda_{\{1,2\}} = p_0^2 p_3^2 + p_0^3 p_2 + p_0^2 p_1 p_3 + p_0 p_1 p_2 p_3$$

$$\mathcal{N}_{\mathbf{p}} : \rho \longmapsto p_0\rho + p_1 X\rho X + p_2 Y\rho Y + p_3 Z\rho Z$$

PROBLEM

Exponential scaling:

$$|V| = n, U \subset V$$

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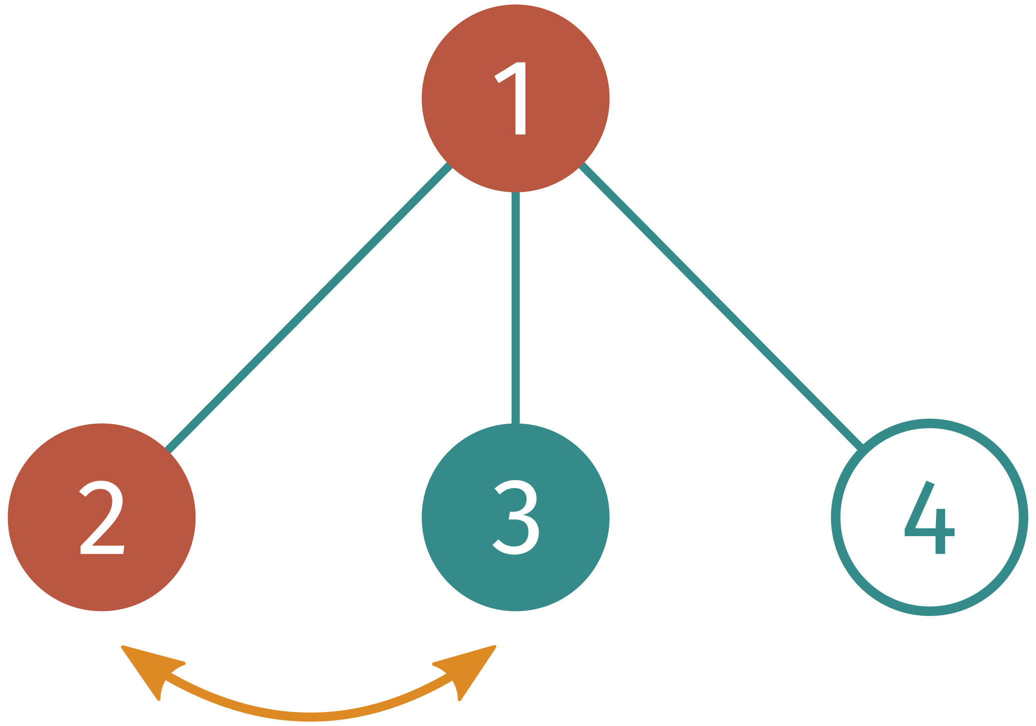
Exploiting graph symmetries

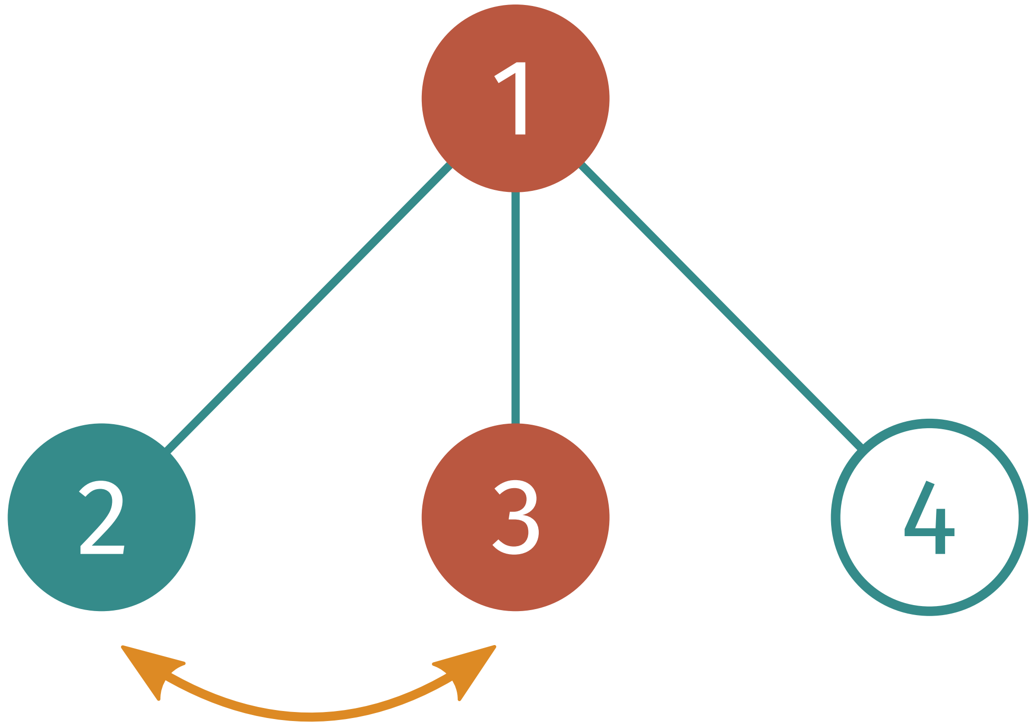
Revisit example: star graph on 4 vertices, $U = \{1, 2\}$.

$$\mathcal{N}^{\otimes n}(|\Gamma\rangle\langle\Gamma|) = \sum_{U \subset V} \lambda_U |U\rangle\langle U|$$



Graph automorphism

$2 \rightleftharpoons 3$



$Z^1Z^2 \quad Y^2 \quad Z^2X^3 \quad Z^1Y^2X^3$

$Z^1Z^3 \quad Y^3 \quad Z^3X^2 \quad Z^1Y^3X^2$

$$p_0^2p_3^2 + p_0^3p_2 + p_0^2p_1p_3 + p_0p_1p_2p_3 = \lambda_{\{1,2\}} = \lambda_{\{1,3\}} = p_0^2p_3^2 + p_0^3p_2 + p_0^2p_1p_3 + p_0p_1p_2p_3$$

Exploiting graph symmetries

Let $G = \mathrm{Aut}(\Gamma, |A|)$ be the **2-colored graph automorphism group** of the graph Γ.

Identify Pauli operators with

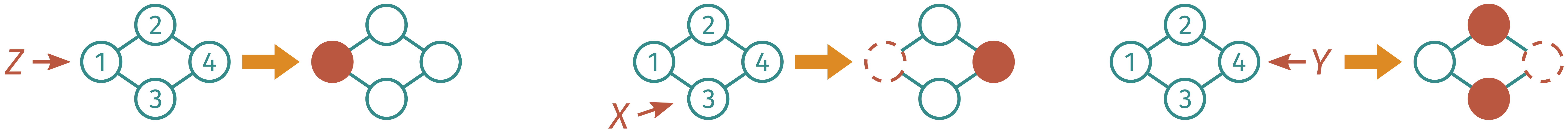
quaternary strings $Q = \{0, 1, 2, 3\}^n$.

Example: $Z^1 X^3 Y^4 \longleftrightarrow \texttt{3012}$

Identify subsets $U \subset V$

with **binary strings** $B = \{0, 1\}^n$.

Example: $U = \{2, 3\} \longleftrightarrow \texttt{0110}$



Group action of G on Q and B by permuting strings is **homomorphic**:

Decoherence rules induce G-equivariant surjective map $\theta : Q \to B$.

$\theta(\texttt{3012}) = \texttt{0110}, \pi = (23)$

$\theta(\pi(\texttt{3012})) = \theta(\texttt{3102}) = \texttt{0110} = \pi(\texttt{0110})$

Exploiting graph symmetries

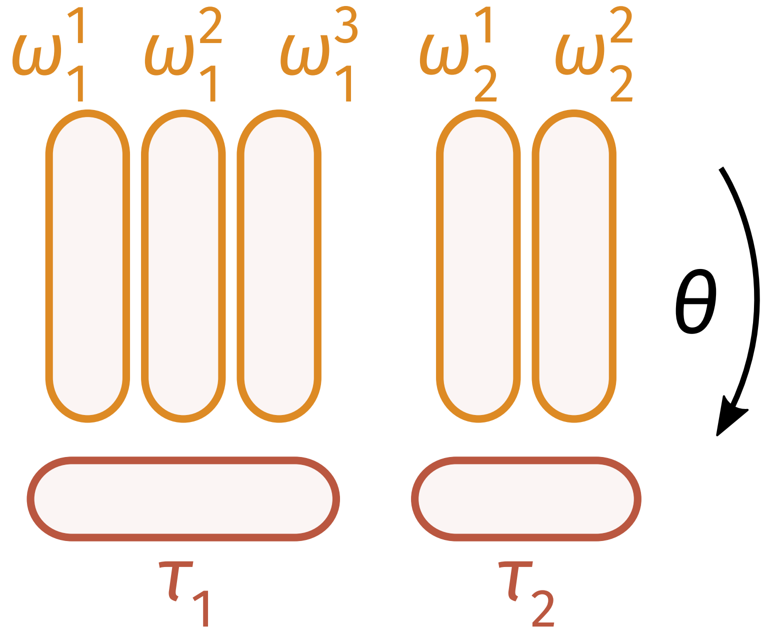
$Q = \{0,1,2,3\}^n, B = \{0,1\}^n$, automorphism group G acts on Q, B by permuting strings.

G-equivariant surjective map $\theta : Q \to B$ defined by decoherence rules.

Homomorphic group actions

→ For each orbit $\omega \in Q/G$ there exists a unique $\tau \in B/G$ such that $\omega \cap \theta^{-1}(\tau) \neq \{\}$.

→ For an orbit $\tau \in B/G$ and subsets $U, U' \in \tau$ we have $|\theta^{-1}(U)| = |\theta^{-1}(U')|$.



To compute $\mathcal{N}^{\otimes n}(|\Gamma\rangle\langle\Gamma|) = \sum_{U \subset V} \lambda_U |U\rangle\langle U|$

and partial trace $\mathrm{Tr}_R\, \mathcal{N}^{\otimes n}(|\Gamma\rangle\langle\Gamma|)$

Symmetry-aware algorithm

Loop over orbit representatives of Q/G

and collect contributions and multiplicities to get λ_U.

Yields **analytical expression** for $\mathcal{I}_c(\Gamma, \mathcal{N}^{\otimes n})$ in the p_i's.

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Numerical method

Given: Pauli channel $\mathcal{N}_{\mathrm{p}_x}$, where $\mathrm{p}_x = (1 - x, xp_1, xp_2, xp_3)$, and a graph state $|\Gamma\rangle$ on $n + 1$ qubits.

Goal: compute coherent information $\mathcal{I}_c(\Gamma, \mathcal{N}^{\otimes n})$ from the output state

$$\mathcal{N}^{\otimes n}(|\Gamma\rangle\langle\Gamma|) = \sum_{U \subset V} \lambda_U |U\rangle\langle U|.$$

Our algorithm yields an **analytical expression** of the coherent information

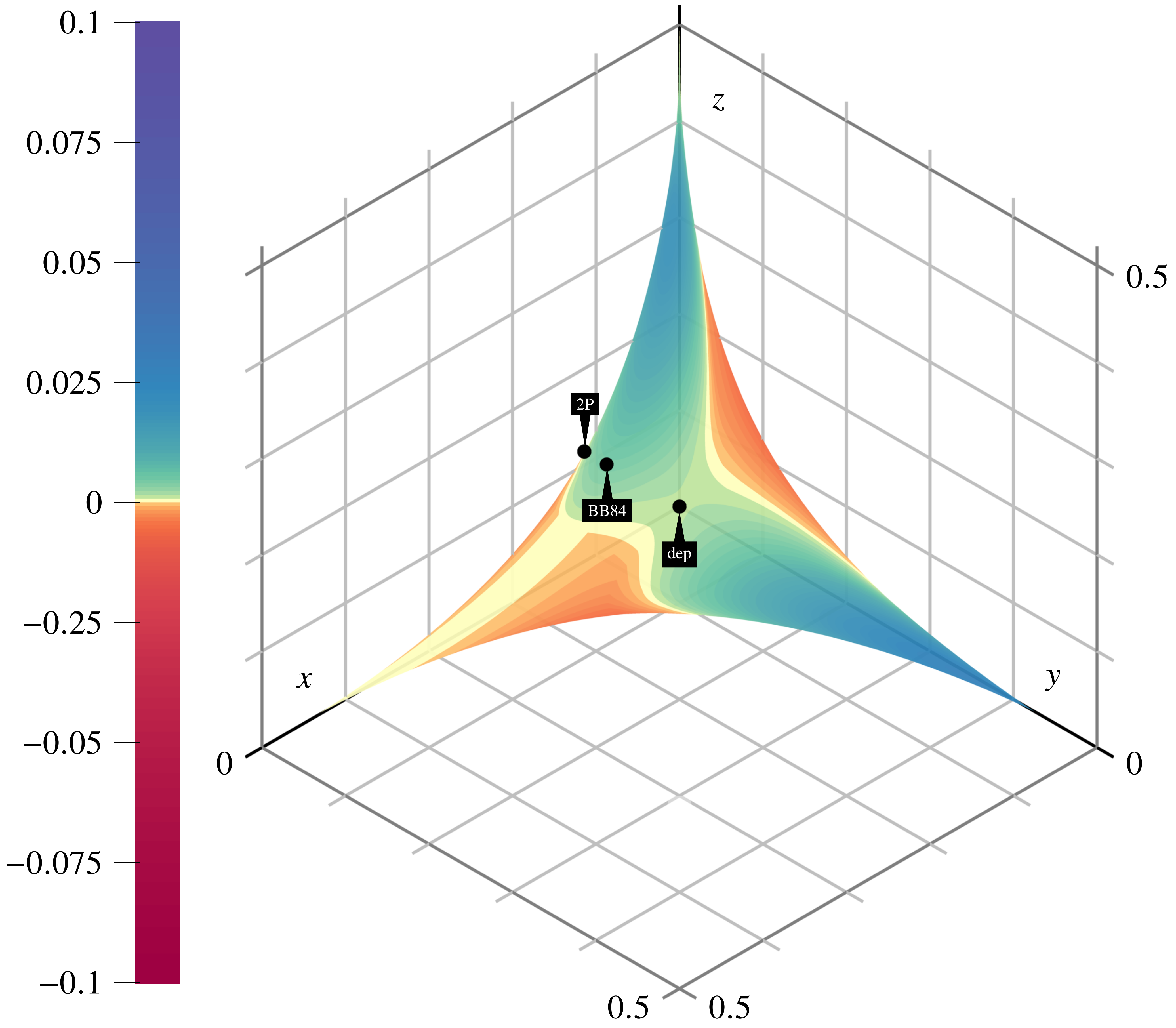
in terms of Pauli probabilities $(1 - x, xp_1, xp_2, xp_3)$, i.e., a function $f(x)$.

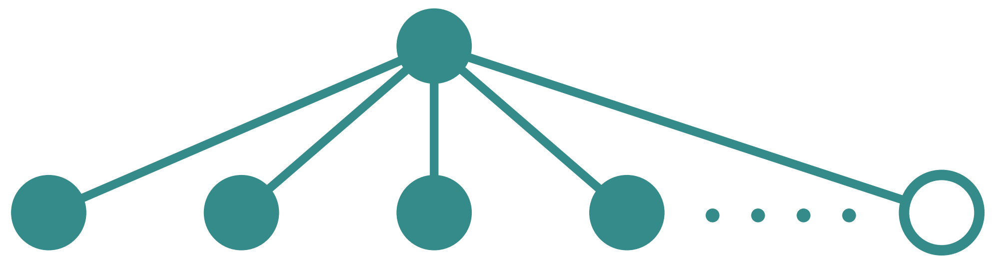
This function $f(x)$ is **non-increasing** in x, so the **error threshold** is given by the **first root of** f.

We can determine this threshold for a fixed graph state in the **whole Pauli channel simplex**

by varying (p_1, p_2, p_3), leading to a **threshold surface**.

Repetition code thresholds





Plot of the **threshold surface** of rep codes with $n \leq 60$.

$x \mapsto \mathbf{p}_x = (1 - x, xp_1, xp_2, xp_3)$

Color: Superadditivity magnitude

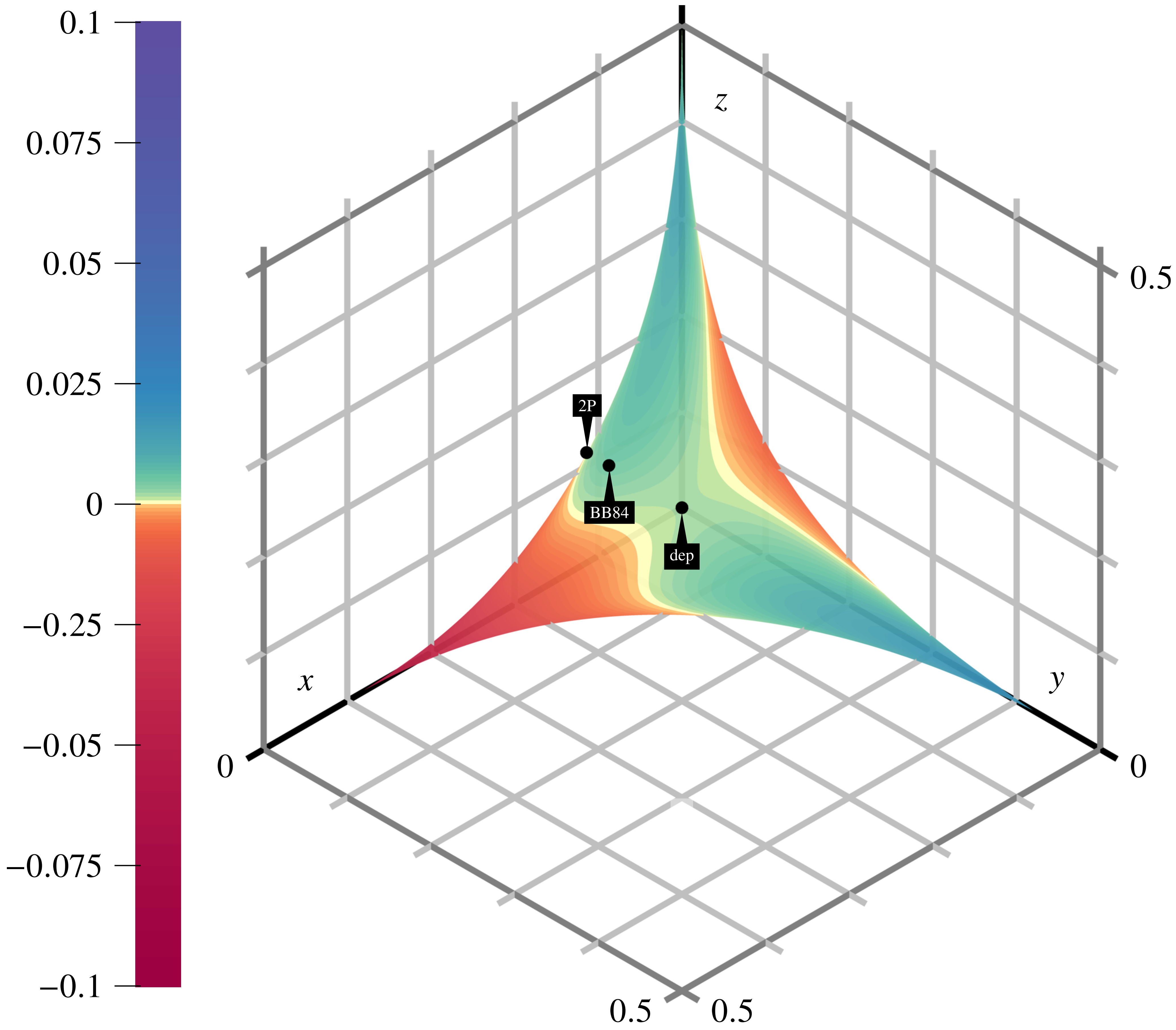
$$\frac{1}{n}\mathcal{I}_c(\Gamma, \mathcal{N}_{\mathbf{p}_x}^{\otimes n}) - \max_{\psi} \mathcal{I}_c(\psi, \mathcal{N}_{\mathbf{p}_x})$$

2P: $(1 - p, \frac{p}{2}, 0, \frac{p}{2})$

BB84: $((1 - p)^2, p - p^2, p^2, p - p^2)$

dep: $(1 - p, \frac{p}{3}, \frac{p}{3}, \frac{p}{3})$

Concatenated code thresholds



5-in-5 code

(graph state on the right)

Achieves best threshold

for depolarizing channel

for $n \leq 25$ channel copies.

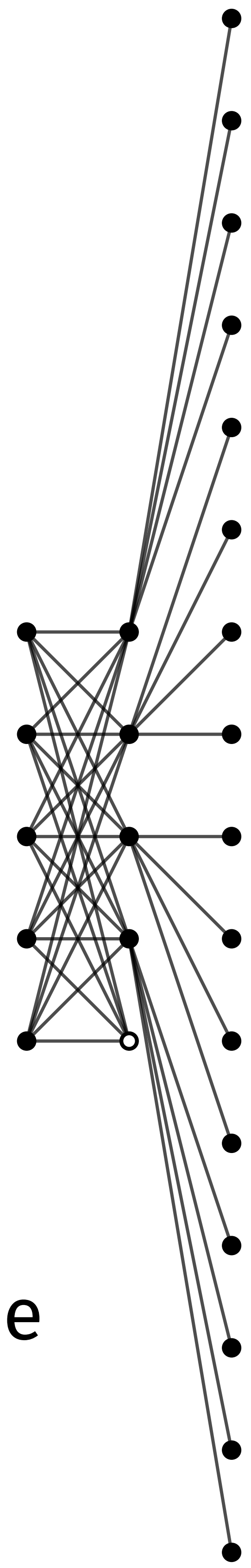
[DiVincenzo et al. '98]

[Smith, Smolin '07; Fern, Whaley '08]

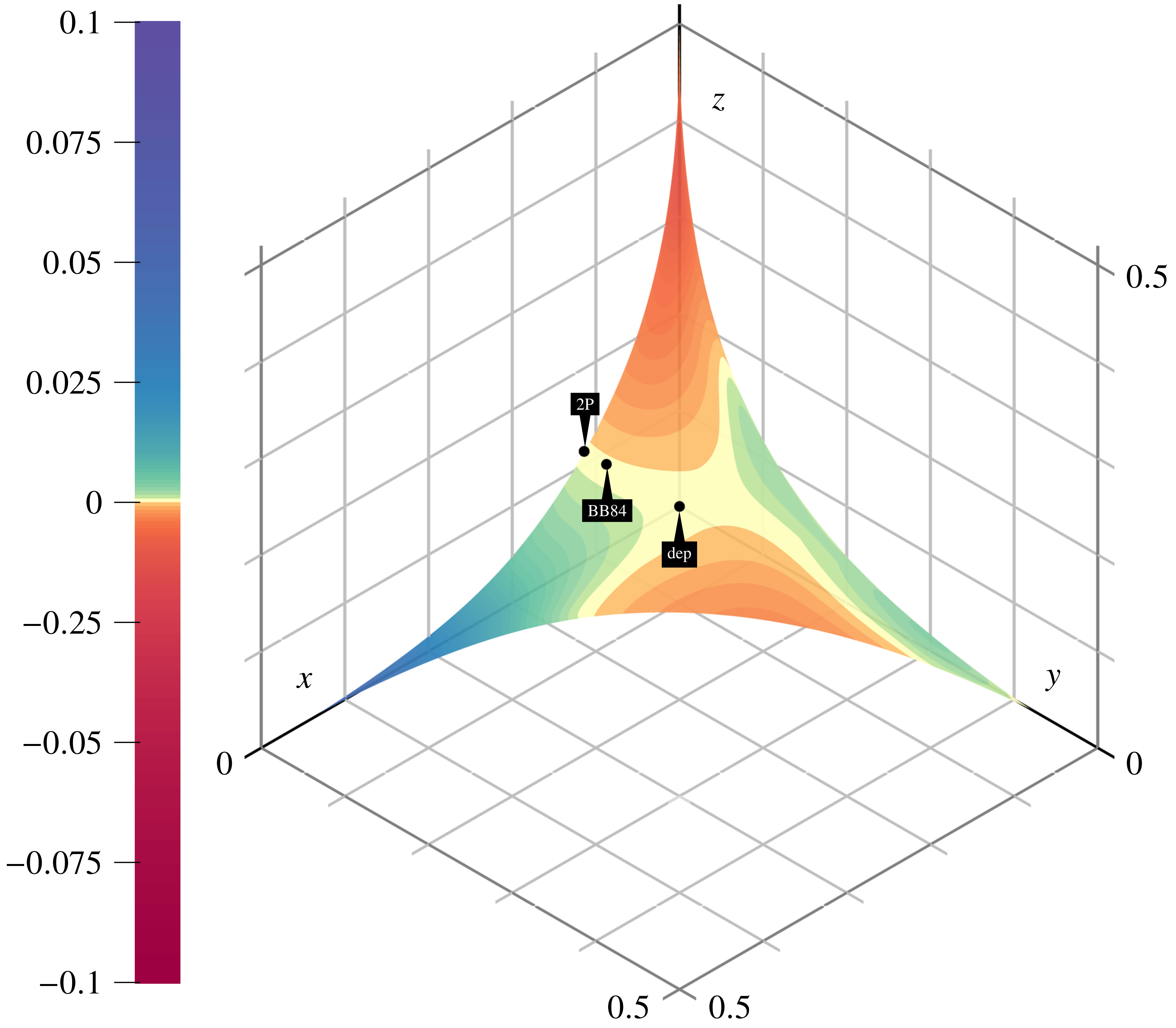
$$x \mapsto \mathbf{p}_x = (1 - x, xp_1, xp_2, xp_3)$$

Color: Superadditivity magnitude

$$\frac{1}{n}\mathcal{I}_c(\Gamma, \mathcal{N}_{\mathbf{p}_x}^{\otimes n}) - \max_{\psi} \mathcal{I}_c(\psi, \mathcal{N}_{\mathbf{p}_x})$$

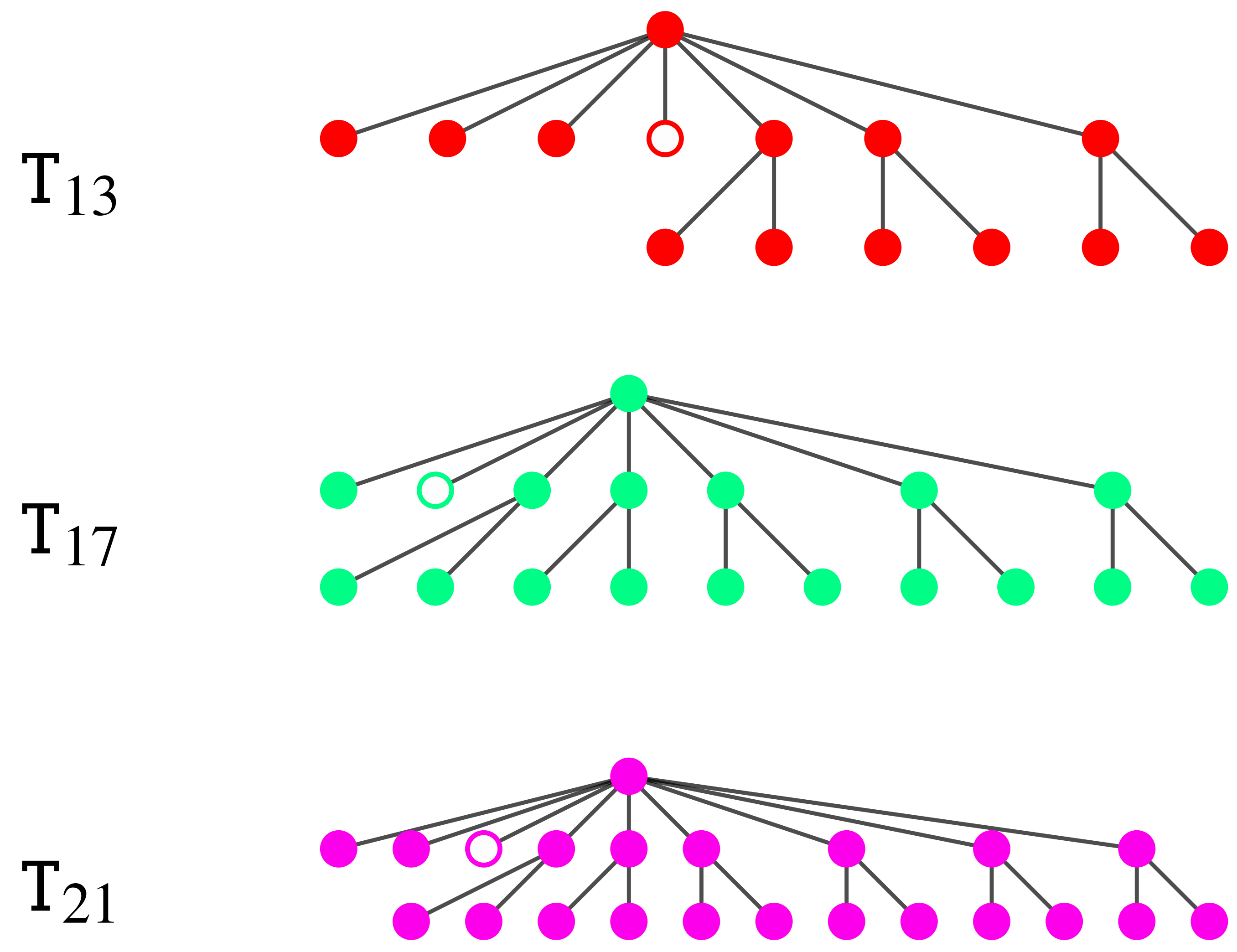


A new code family: tree graph states



We found a new interesting code family based on **tree graphs with two levels**.

Error thresholds are competitive compared to 5-in-5 code (see left).



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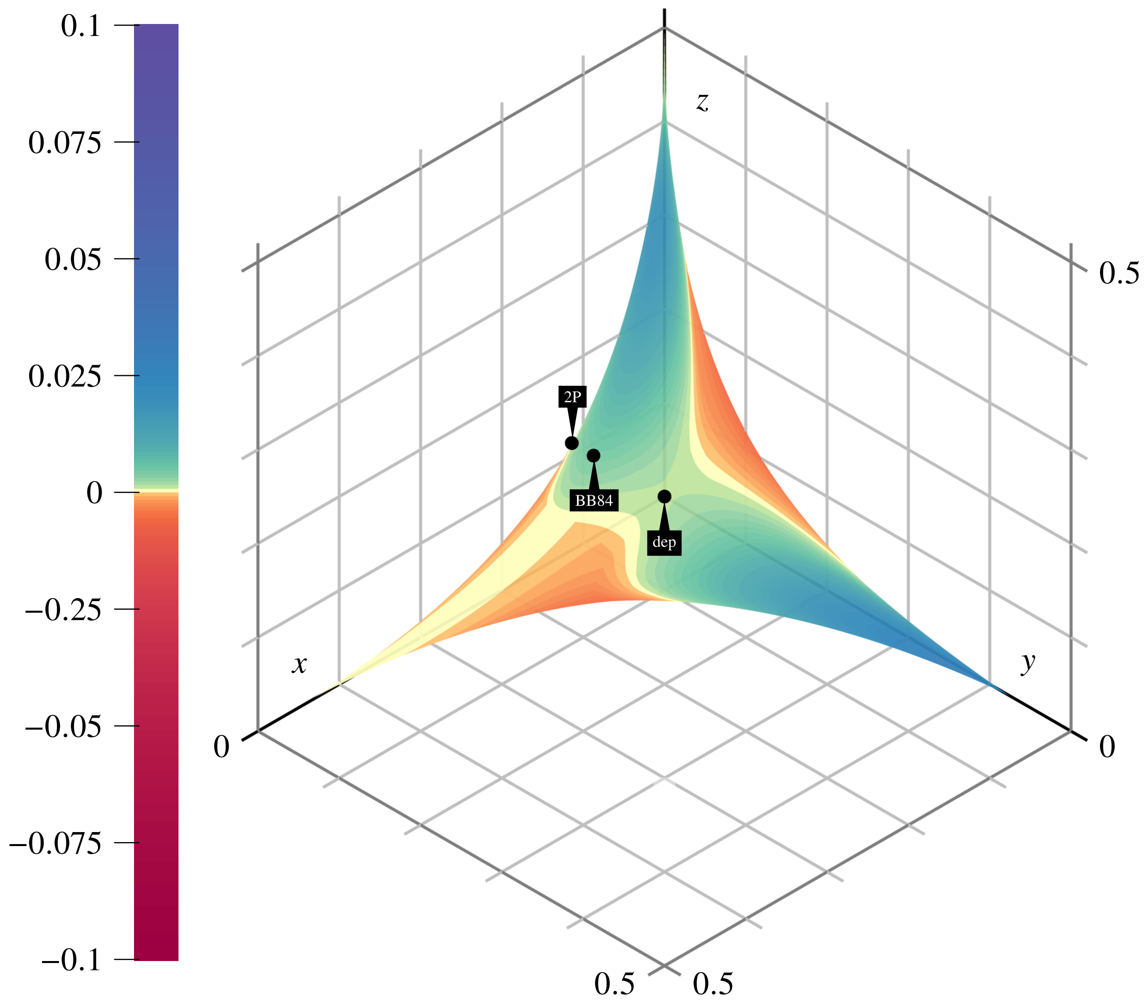
Symmetries in channel coding problems

Summary

We can use the graph state formalism and exploit **graph symmetries** and tools from **group theory** to approximate quantum capacity of interesting channels.



Ideas for future work:

➡ Generalize methods to handle more general quantum channels?

➡ Improve upon/adopt more tools from computational group theory (CGT)?

➡ Can we use the framework of group actions and CGT to analyze concrete quantum error correction codes and their decoders, thresholds, etc?

Our paper
on arXiv.org

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these slides

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THANKS

for your attention!