Port-based teleportation in arbitrary dimension

Asymptotics and a converse bound

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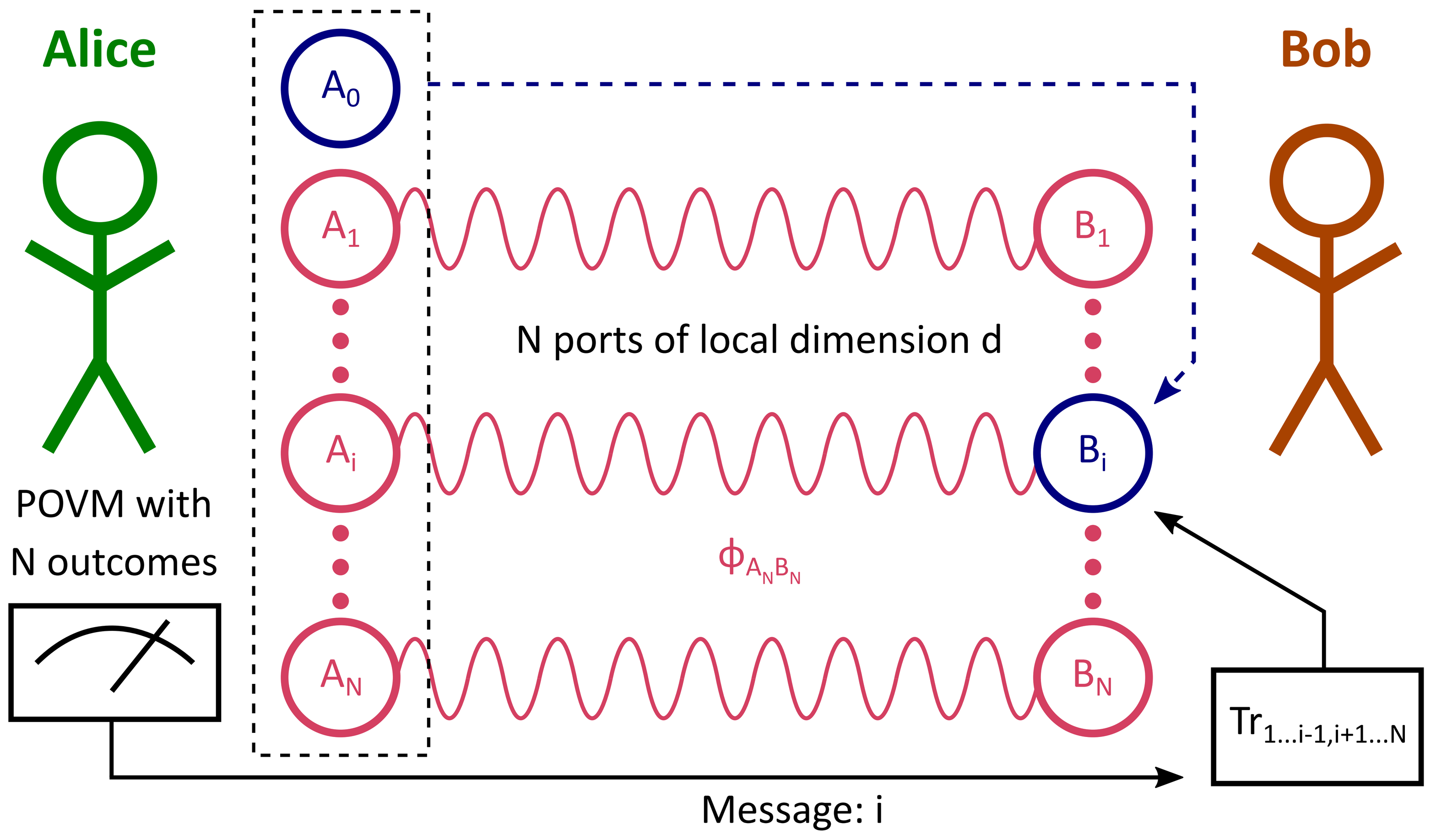
Port-based teleportation

- Port-based teleportation (PBT) was introduced by Ishizaka & Hiroshima [1] as a **unitarily covariant** version of teleportation.
- PBT enables instantaneous non-local quantum computation [2], which can be used to break position-based cryptography [3].
- Caveat: Because of unitary covariance, **perfect PBT is impossible** with finite resources [1].



Variants of PBT

- **Deterministic PBT** (dPBT): Protocol yields final state that approximates target state, implementing a d-dimensional channel $\Lambda \approx \mathrm{Id}_d$.
- Figure of merit: entanglement fidelity $F_d = F(\Lambda, \mathrm{Id}_d)$.
- Unitary covariance: entanglement fidelity is equivalent to diamond norm.
- **Probabilistic PBT** (pPBT): Protocol yields exact target state with certain success probability p_d, otherwise aborts.

Standard protocols

- dPBT is equivalent to state discrimination of the uniformly drawn states $\omega^{(i)}_{A^N B} = \mathrm{Tr}_{B_{i^c}} \Phi_{A^N B^N}$ [4].
- This link suggests using **pretty good measurement** (PGM) as POVM.
- Further protocol simplification: use N EPR pairs as ports.
- Write F_d^{std} for dPBT with PGM and EPR, p_d^{std} for pPBT with EPR.

Symmetries

- PBT has **inherent symmetries** such as permutation invariance (S_N) of Bob's port systems B_i or Alice's POVM elements, and unitary invariance (U_d).
- S_N and U_d have natural commuting representations on $(\mathbb{C}^d)^{\otimes N}$.
- **Schur-Weyl decomposition:** $(\mathbb{C}^d)^{\otimes N} \cong \bigoplus_{\lambda \vdash_d N} [\lambda] \otimes V_\lambda$. ($\lambda$ Young diagram.)
- $[\lambda]$ is an irrep of S_N with $\dim[\lambda] = d_\lambda$, and V_λ is an irrep of U_d with $\dim V_\lambda = m_\lambda$.

Exact formulas for standard protocols [5,6]

- $$F_d^{\mathrm{std}} = \frac{1}{d^{N-2}} \sum_{\alpha \vdash_d N-1} \left(\sum_{\mu = \alpha + \square} \sqrt{d_\mu m_\mu} \right)^2, \qquad (1)$$
  where $\mu = \alpha + \square$ means adding a box to a Young diagram $\alpha \vdash_d N-1$.
- $$p_d^{\mathrm{std}} = \frac{1}{d^N} \sum_{\alpha \vdash_d N-1} m_\alpha^2 \frac{d_{\mu^*}}{m_{\mu^*}}, \qquad (2)$$
  where $\alpha \vdash_d N-1$ and $\mu^* = \alpha + \square^*$ is such that $N \frac{m_\mu d_\alpha}{m_\alpha d_\mu}$ is maximal.

Main result 1: Asymptotics of PBT

- For deterministic PBT using PGM and EPR, we prove that
  $$F_d^{\mathrm{std}} = 1 - \frac{d^2-1}{4N} + o(N^{-1}). \qquad (3)$$
- For probabilistic PBT using EPR, we prove that
  $$p_d^{\mathrm{std}} = 1 - \sqrt{\frac{d}{N-1}} \mathbb{E}[\lambda_{\max}(\mathbf{G})] + o(N^{-1}), \qquad (4)$$
  where $\mathbf{G} \sim \mathrm{GUE}_0(d)$ is drawn from the traceless Gaussian unitary ensemble.
- (3) and (4) recover known qubit results derived in [4].
- Proves that known bound $F_d^{\mathrm{std}} \geq 1 - (d^2-1)/N$ [1,2] is not tight.
- For qubits (i.e., $\mathbf{G}$ is a 2×2 matrix), we have $\mathbb{E}[\lambda_{\max}(\mathbf{G})] = 2\pi^{-1/2}$.

Schur-Weyl distribution and spectrum estimation

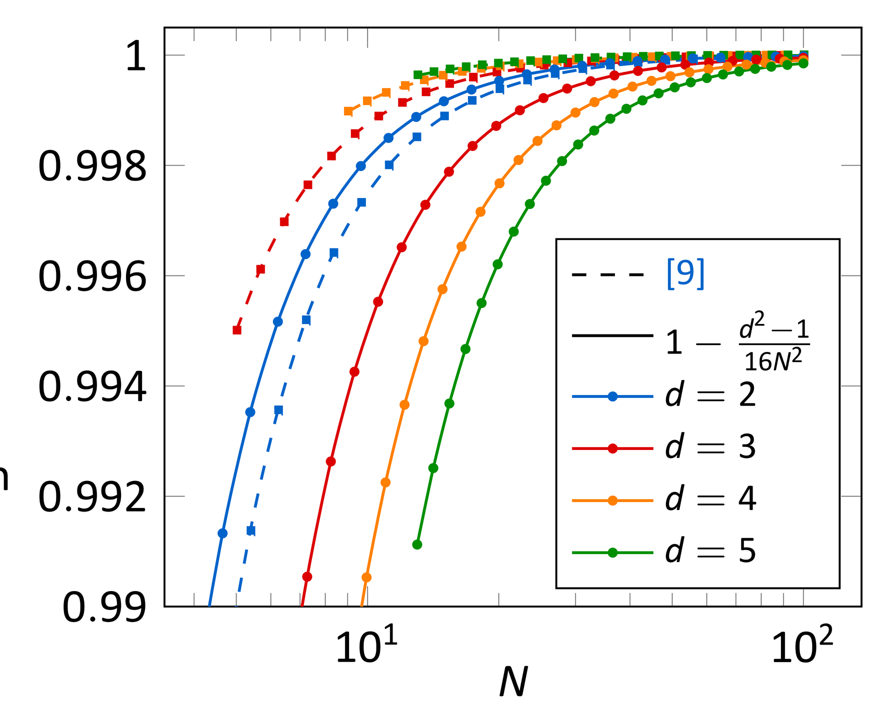
- Let P_λ be the projector onto the summand corresponding to the Young diagram λ in the Schur-Weyl decomposition (see left column).
- Let $p_{d,N} = \frac{d_\lambda m_\lambda}{d^N}$ be the **Schur-Weyl distribution**, corresponding to obtaining outcome λ when applying the measurement $\{P_\lambda\}_{\lambda \vdash_d N}$ to $(\frac{1}{d} I_d)^{\otimes N}$.
- Spectrum estimation [7]: $\frac{1}{N}\mathbf{Y}_N \xrightarrow{D} (1/d, \ldots, 1/d)$ in distribution for $\mathbf{Y}_N \sim p_{d,N}$.

Fluctuations of the Schur-Weyl distribution

- A "typical" Young diagram w.r.t. $p_{d,N}$ has $\lambda_1 \approx \frac{N}{d} + 2\sqrt{N}$ and $\lambda_d \approx \frac{N}{d} - 2\sqrt{N}$, and the remaining λ_i's interpolate between these.
- Go to centered and normalized random variable $\mathbf{A}_N = \frac{\lambda_N - (N/d, \ldots, N/d)}{\sqrt{N/d}}$.
- "Central limit theorem" [8]: $\mathbf{A}_N \xrightarrow{D} \mathrm{spec}(\mathbf{G})$ in distribution, where $\mathbf{G} \sim \mathrm{GUE}_0(d)$ is drawn from the traceless Gaussian unitary ensemble.
- Proof idea for (3) and (4): Rewrite (1) and (2) as expectation values over Schur-Weyl distribution $p_{d,N}$ and then use a strengthened version of the above.

Main result 2: Converse bound (non-asymptotic)

- For an arbitrary PBT protocol,
  $$F_d \leq \begin{cases} \frac{\sqrt{N}}{d} & \text{if } N \leq d^2/2, \\ 1 - \frac{d^2-1}{16N^2} & \text{else.} \end{cases}$$
- Tighter than previously known bound $1 - \frac{1}{4(d-1)N^2}$ from [9].
- Based on a no-signaling argument.



Main result 3: dPBT with fully optimized POVM and entangled state

- Entanglement fidelity F_d^* also has exact formula [6], with an optimized probability distribution $q_{d,N}$ replacing $p_{d,N}$.
- We derive the following bound, matching the $O(N^{-2})$-dep. in **Main result 2**:
  $$F_d^* \geq 1 - \frac{d^5 + O(d^{9/2})}{4\sqrt{2}N^2} + O(N^{-3}). \qquad (5)$$
- Proof idea: Pick as $q_{d,N}$ a sufficiently smooth probability distribution on the ordered probability distribution simplex OS_{d-1}.
- $F_d(q_{d,N})$ can be expressed in terms of the first Dirichlet eigenvalue of the Laplacian on OS_{d-1}, which can be bounded using methods from functional analysis.

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