TQC 2020 June 10, 2020

Playing Games with Multiple Access Channels

Nature Communications **11**, 1497 (2020) arXiv:1909.02479

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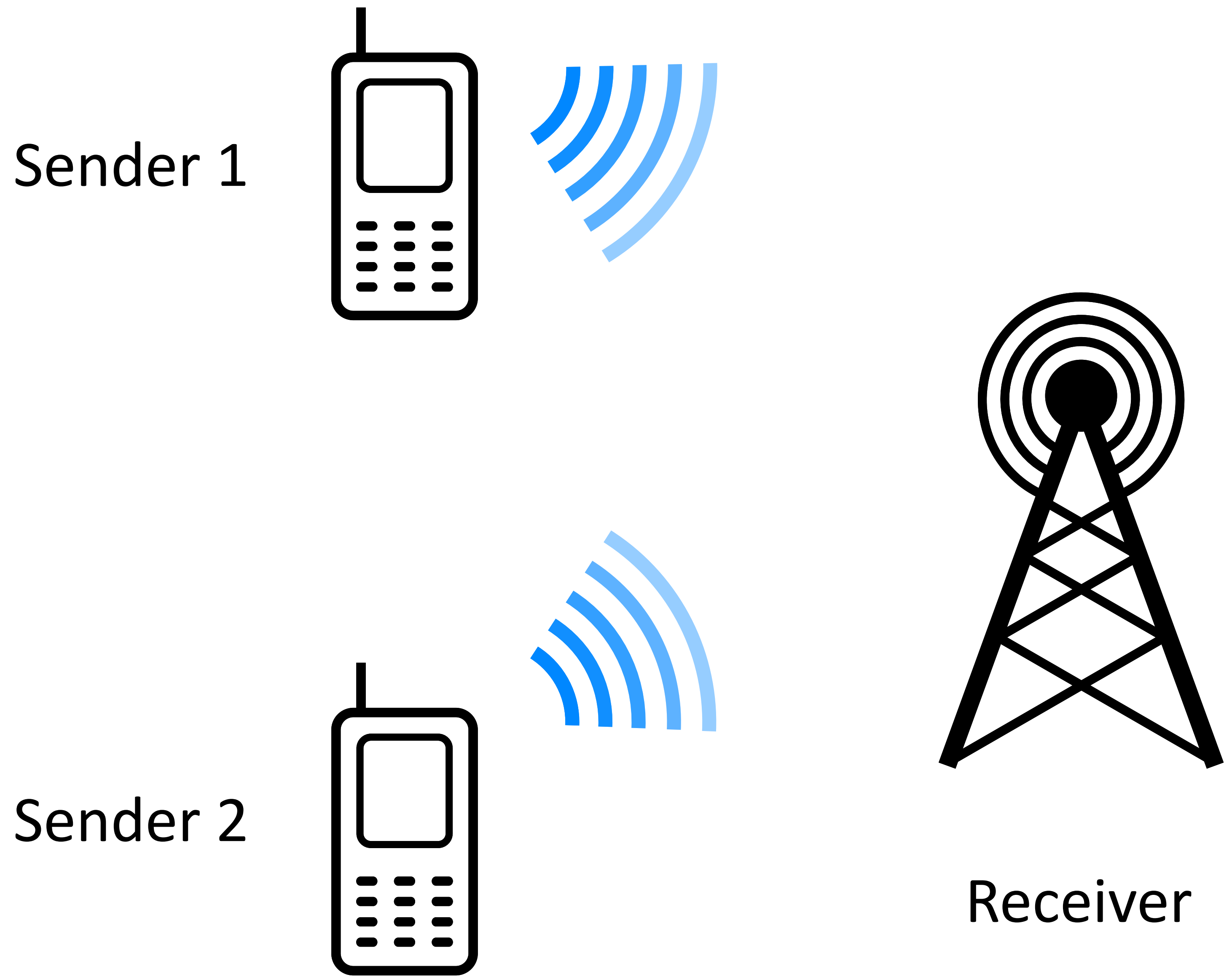
Multiple access channel

Simplest network communication scenario involving two senders and one receiver.



Goal

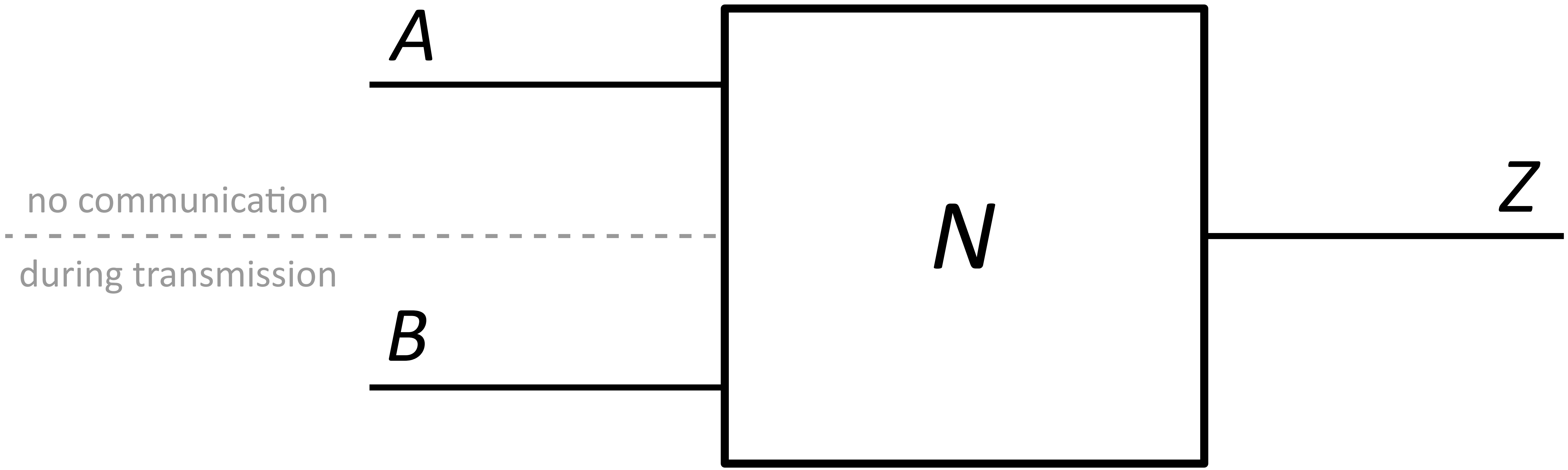
Each sender transmits individual classical messages through common channel to the receiver.

Multiple access channel

Input RVs A (sender 1) and B (sender 2).

MAC: Conditional probability distribution $N(z|a,b)$ defines output RV Z.

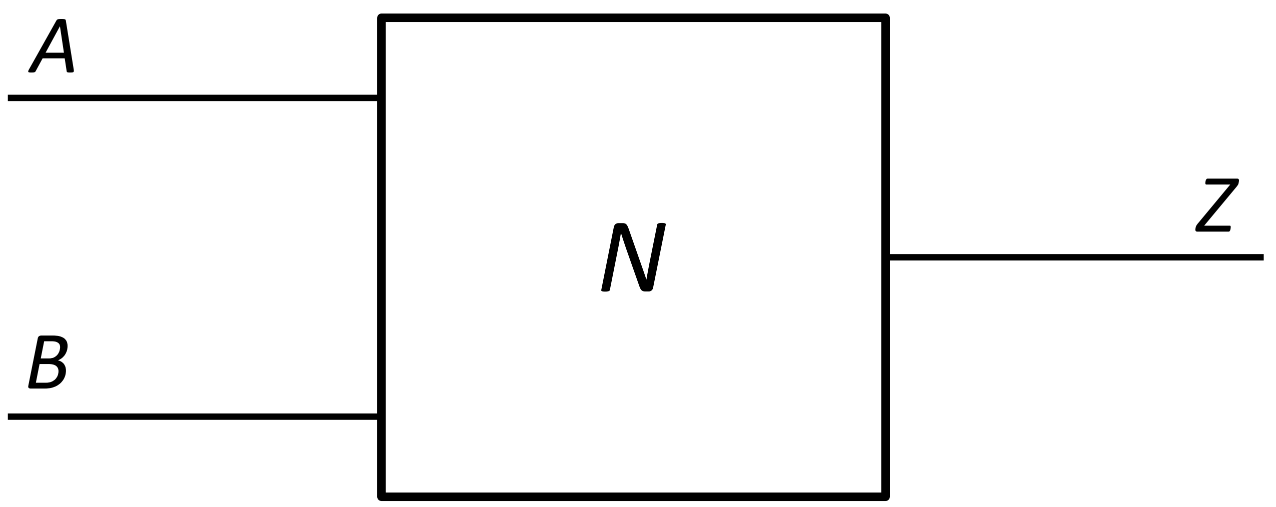
No communication between senders: A, B are **independent**.



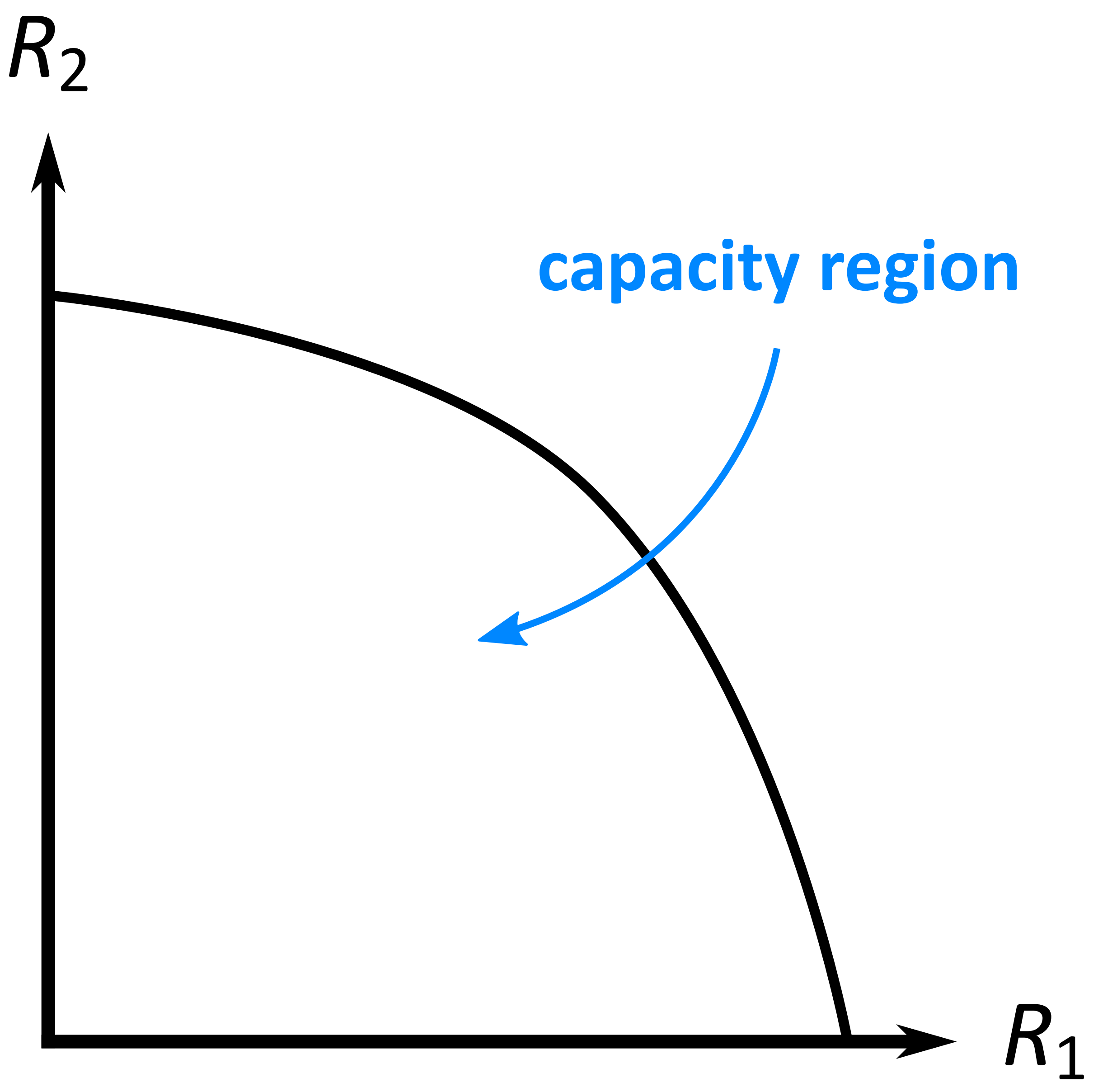
Capacity region of a MAC

Sender 1 (2) tries to send information at rate R_1 (R_2).
(R_1, R_2) is called *achievable* if receiver can decode the two messages with vanishing error.

Capacity region: closure of the set of all achievable rate pairs (R_1, R_2).



Multiple access channel



Typical capacity region

Capacity region of a MAC

Single-letter capacity region of a MAC **(Ahlswede '73, Liao '73)**

For random variables (A, B) with fixed product distribution $p_A(a)p_B(b)$, let Z be the RV defined by the MAC $N(z|a, b)$.

The **capacity region** of N is the convex hull of all (R_1, R_2) satisfying

$$R_1 \leq I(A; Z|B) \qquad R_2 \leq I(B; Z|A) \qquad R_1 + R_2 \leq I(AB; Z),$$

as $p_A p_B$ varies over all product distributions.

Shannon entropy:

$H(X) = -\sum_x p(x) \log p(x)$

Mutual information:

$I(X; Y) = H(X) + H(Y) - H(XY)$

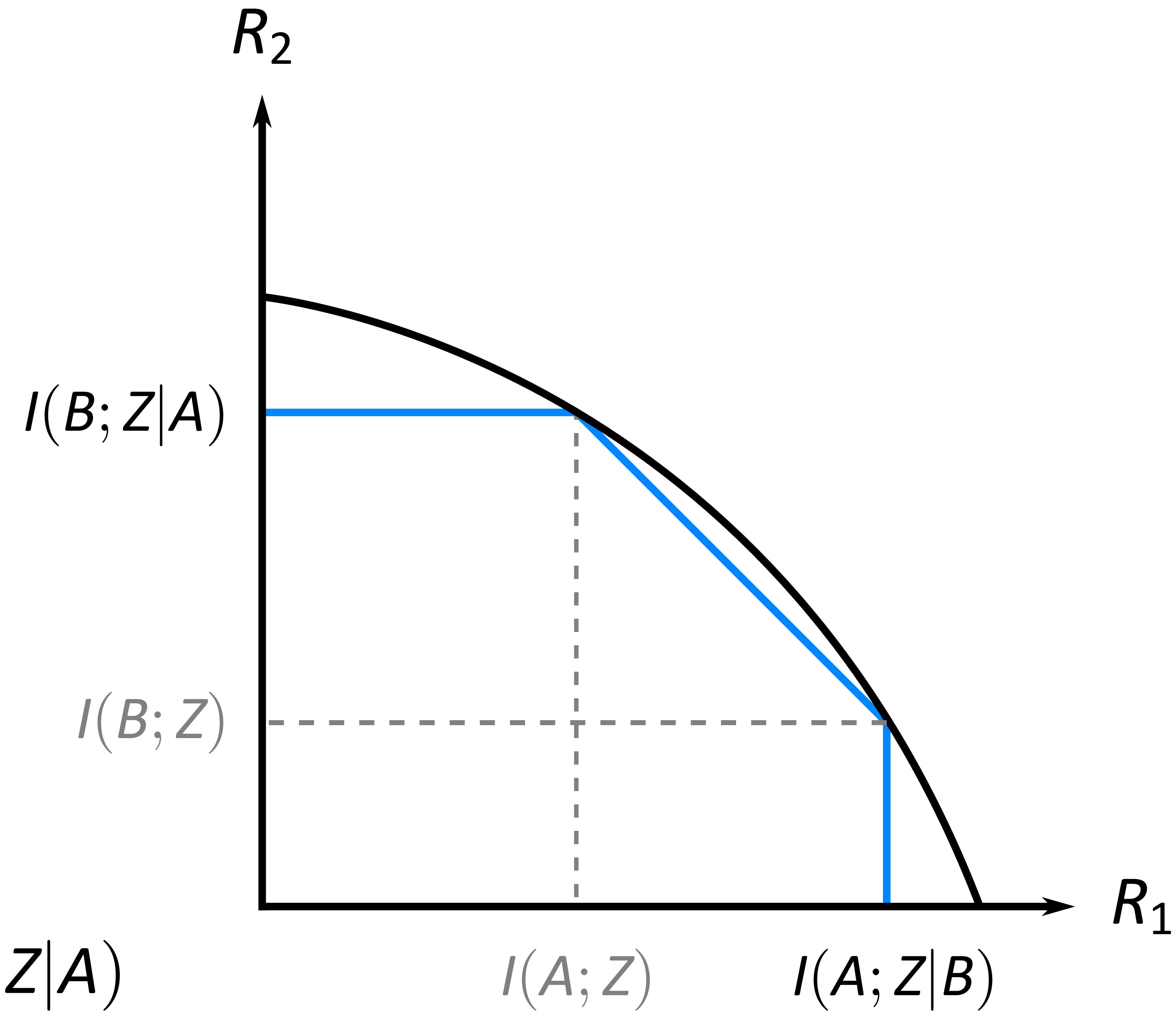
Conditional mutual information:

$I(X; Y|Z) = I(X; YZ) - I(X; Z)$

Typical capacity region of a MAC

Constraints for capacity region $\mathcal{C}$:

$$\begin{aligned} R_1 &\leq I(A;Z|B) \\ R_2 &\leq I(B;Z|A) \\ R_1 + R_2 &\leq I(AB;Z). \end{aligned}$$



For fixed product distribution $p_A p_B$

this region is **pentagonal**, since:

$$\begin{aligned} \max\{I(A;Z|B), I(B;Z|A)\} &\leq I(AB;Z) \\ &\leq I(A;Z|B) + I(B;Z|A) \end{aligned}$$

Capacity region of a MAC

Ahlswede-Liao region characterized by **single-letter formula**.

Complicated part: **product constraint** (←independence constraint) on input RVs.

Question 1

Can we use **entanglement assistance** to boost transmission rates?

YES

Question 2

How hard is it to compute the full region?

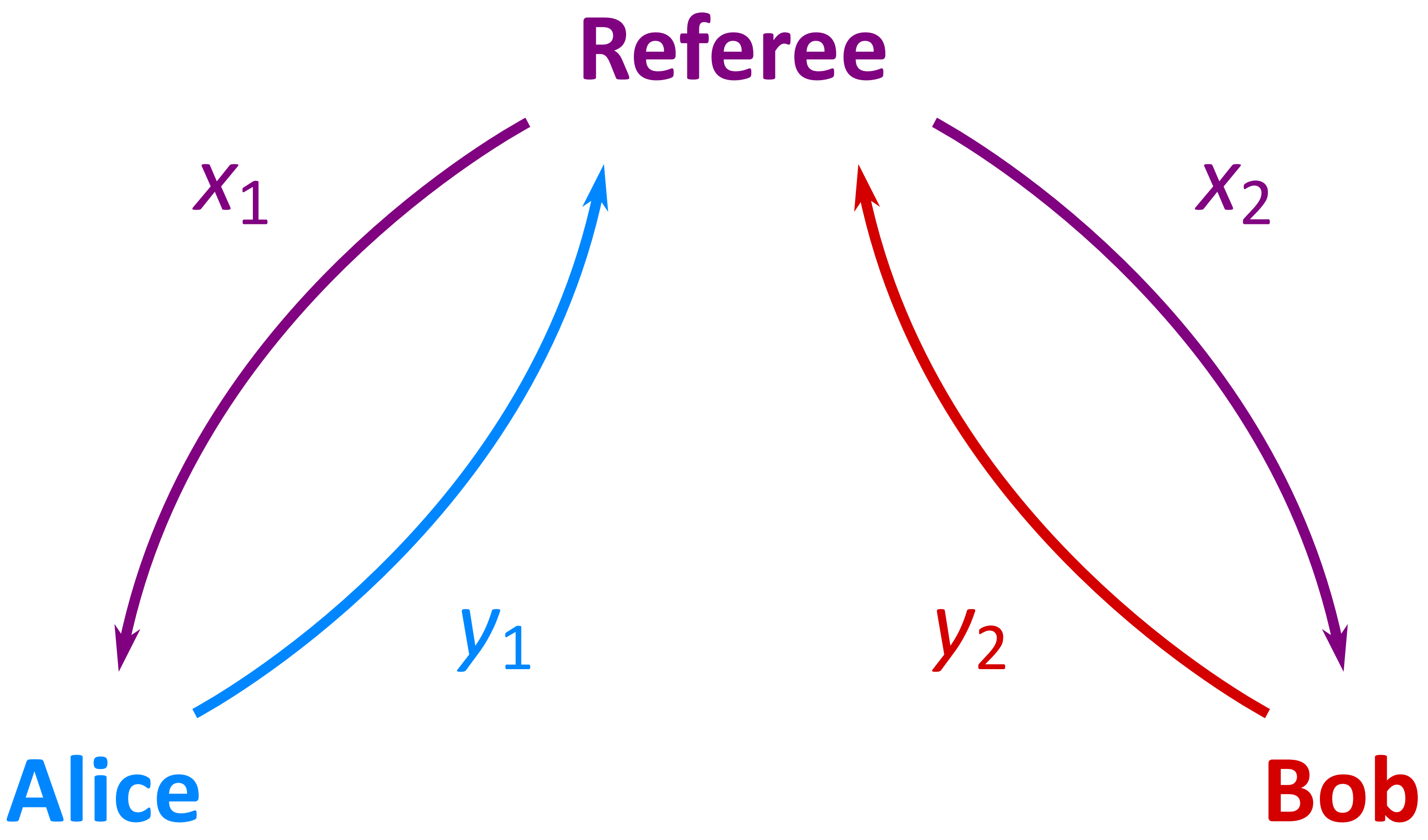
NP-HARD

We will study both questions using the theory of **non-local games**.

For simplicity: focus on the **sum rate** $\max\{R_1 + R_2 : (R_1, R_2) \in \mathcal{C}(N)\}$.

Sum rate constraint: $R_1 + R_2 \leq I(AB; Z)$ for independent A, B.

Non-local games



Questions $x_i \in \mathcal{X}_i$

Answers $y_i \in \mathcal{Y}_i$

Winning condition $W \subseteq \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}_1 \times \mathcal{Y}_2$

Non-local game $G = (\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2, W)$.

Referee draws questions (x_1, x_2) according to some distribution.

Alice answers y_1, Bob answers y_2.

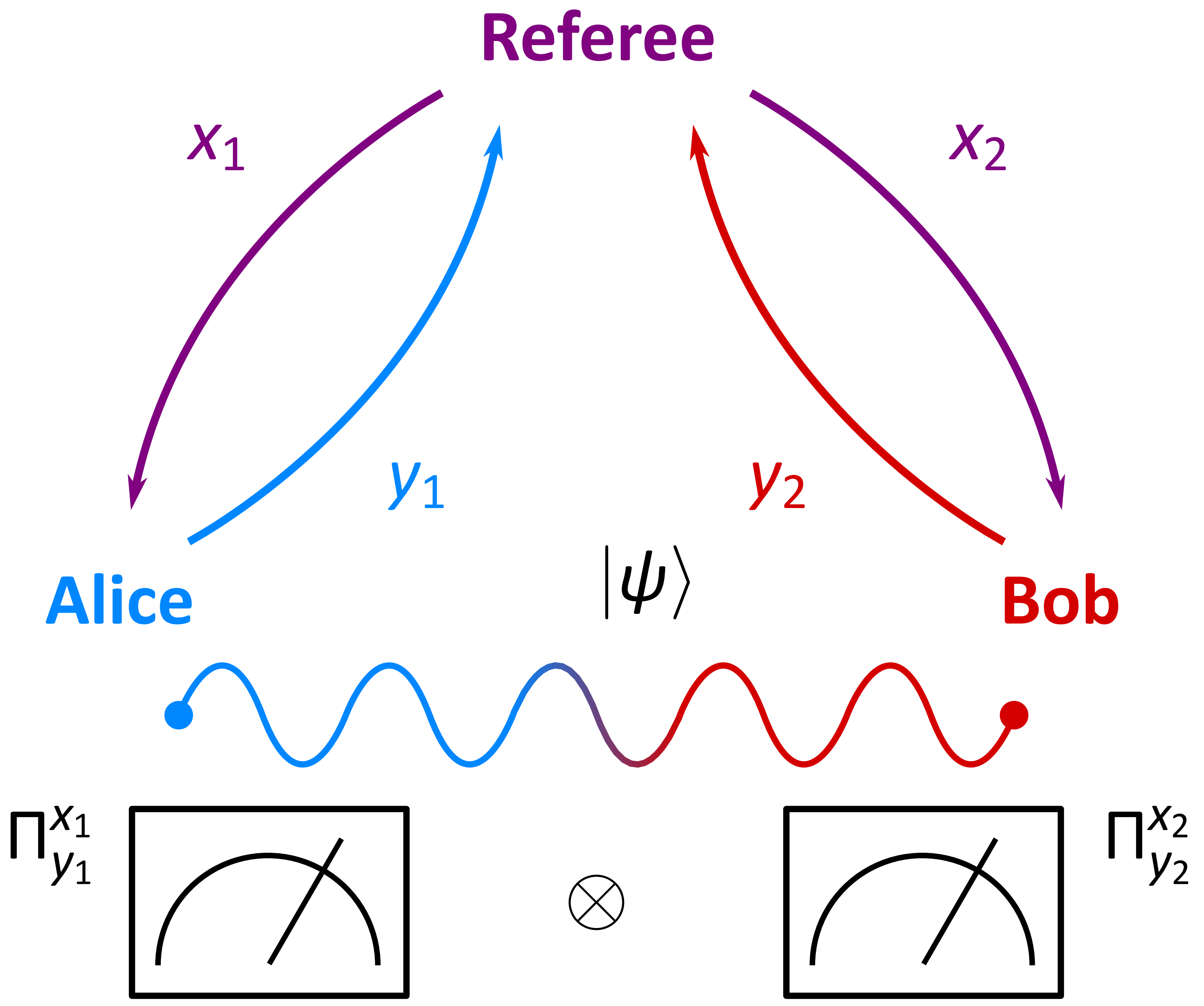
They win if $(x_1, y_1, x_2, y_2) \in W$.

No communication allowed for Alice and Bob to produce answers y_i.

Example: CHSH game

Winning condition: $y_1 \oplus y_2 = x_1 \wedge x_2$

Non-local games: Quantum strategies



$\Pi^{x_i}_{y_i} \geq 0, \quad \sum_{y_i} \Pi^{x_i}_{y_i} = I$

Classical value $\omega(G)$:

Maximal classical winning probability.

Quantum strategies: Alice and Bob measure a shared entangled state $|\psi\rangle$ using POVMs $\{\Pi^{x_i}_{y_i}\}_{y_i \in \mathcal{Y}_i}$.

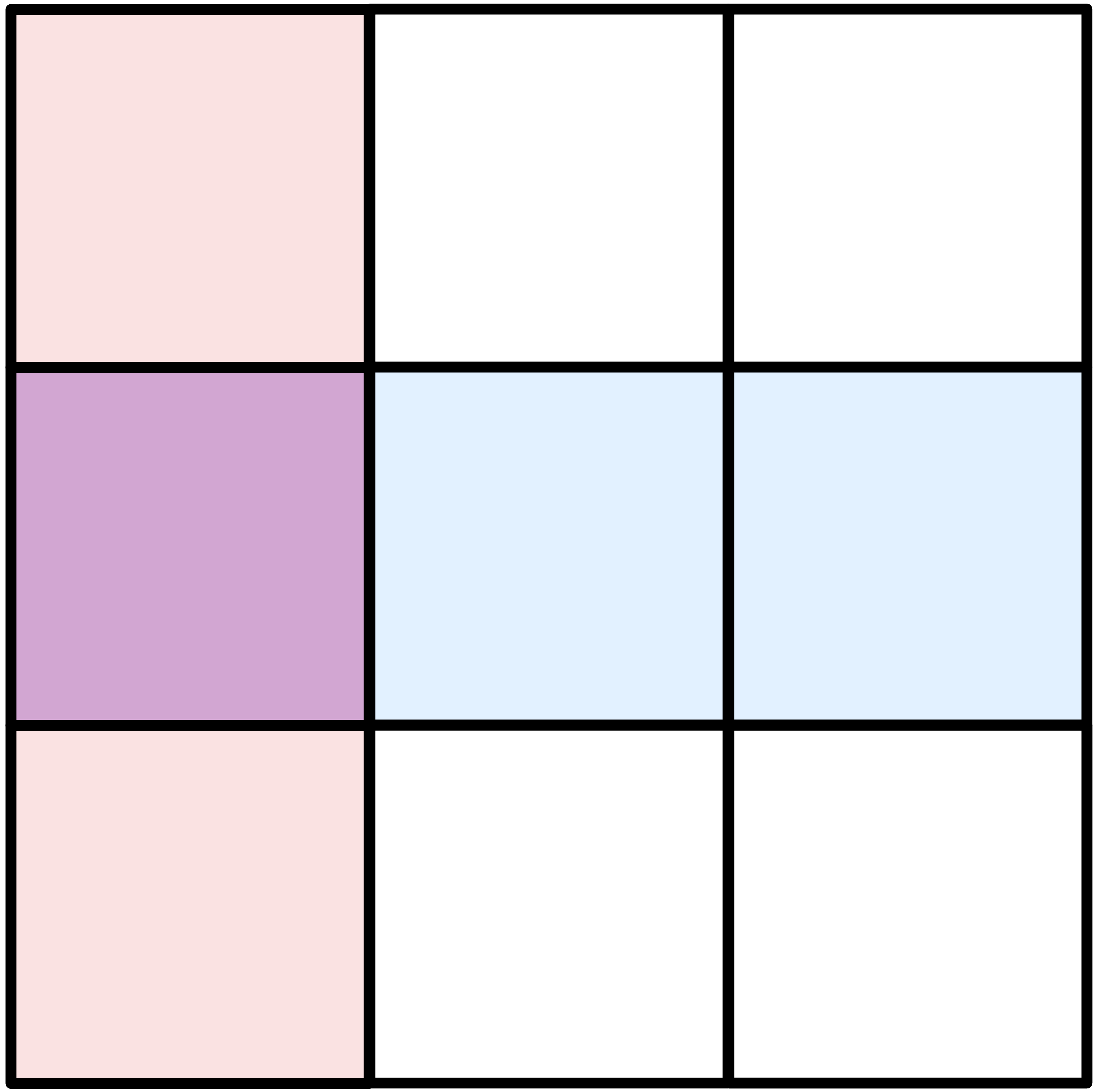
Quantum value $\omega^*(G)$:

maximal quantum winning probability.

Example: CHSH-game G_C

$0.75 = \omega(G_C) < \omega^*(G_C) \approx 0.85$

Magic square game



Alice is given a row.

Bob is given a column.

Both answer with bit strings of length 3.

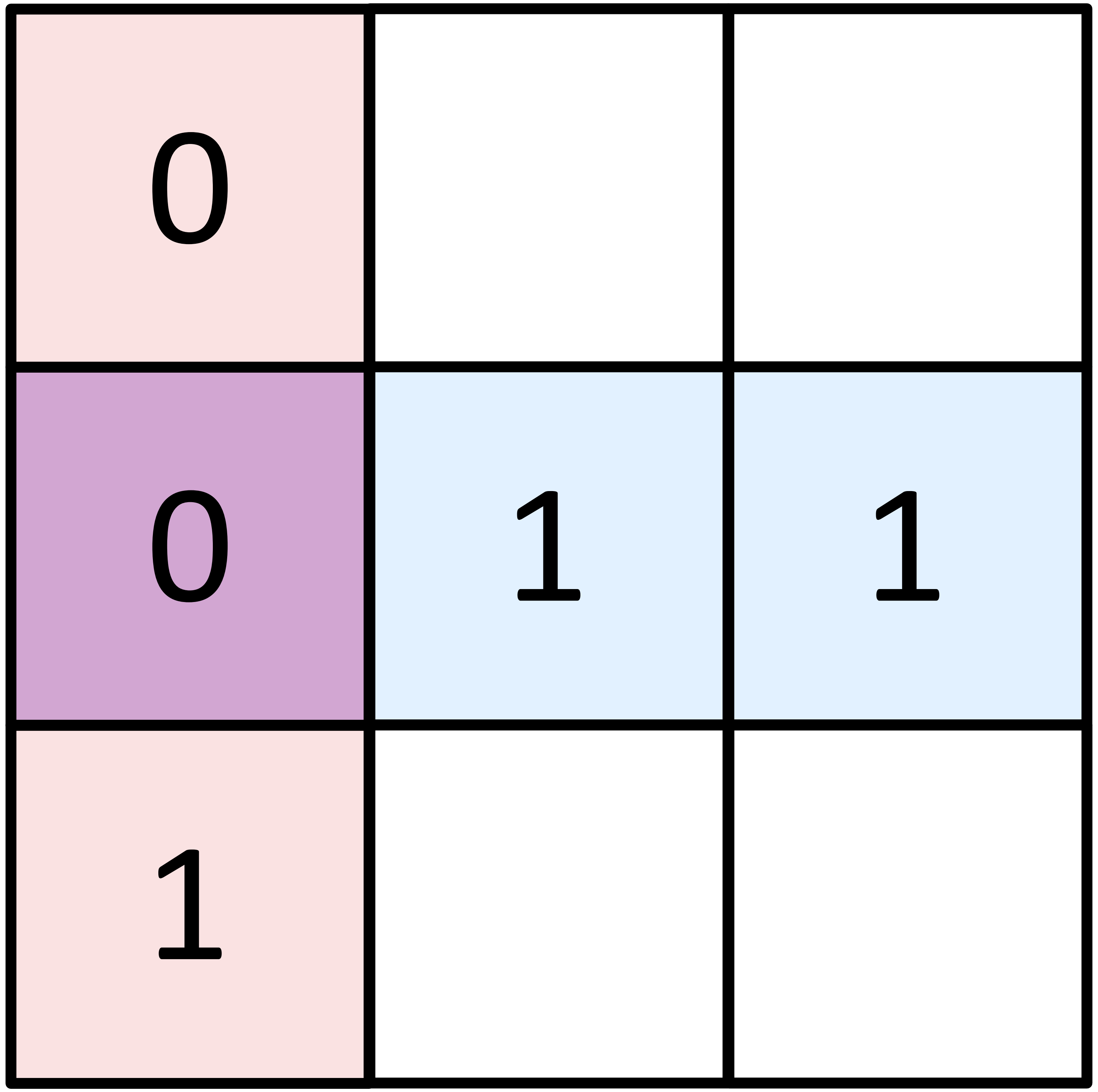
They win, if:

- Alice's parity is even;
- Bob's parity is odd;
- strings agree in overlapping cell.

[Mermin, PRL 65.27 (1990)]

[Peres, Phys. Lett. A 151.3 (1990)]

Magic square game



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Magic square game

$$\frac{1}{2}\left(|00\rangle_{A_1B_1} + |11\rangle_{A_1B_1}\right) \otimes \left(|00\rangle_{A_2B_2} + |11\rangle_{A_2B_2}\right)$$

0	0	0
0	1	1
1	0	?

+XI	+XX	+IX
-XZ	+YY	-ZX
+IZ	+ZZ	+ZI

Classical value

$\omega(G_{MS}) = 8/9$

Quantum value

$\omega^*(G_{MS}) = 1$

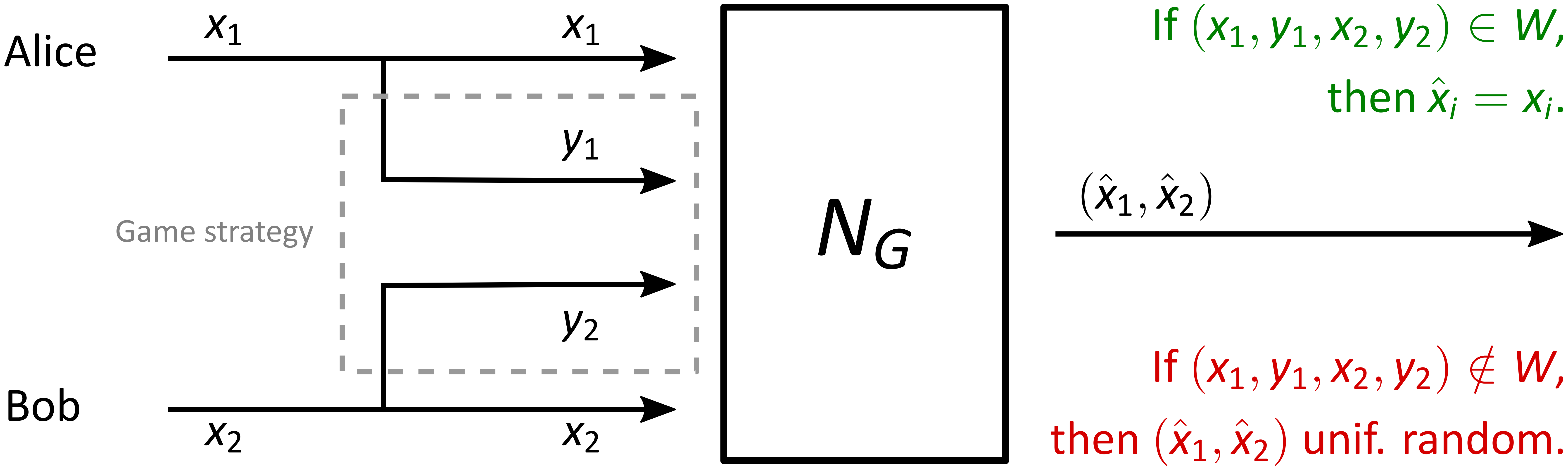
[Mermin, PRL 65.27 (1990)], [Peres, Phys. Lett. A 151.3 (1990)], [Brassard et al., Found. Phys. 35.11 (2005)]

MAC in terms of a non-local game

Let $G = (\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2, W)$ be a non-local game.

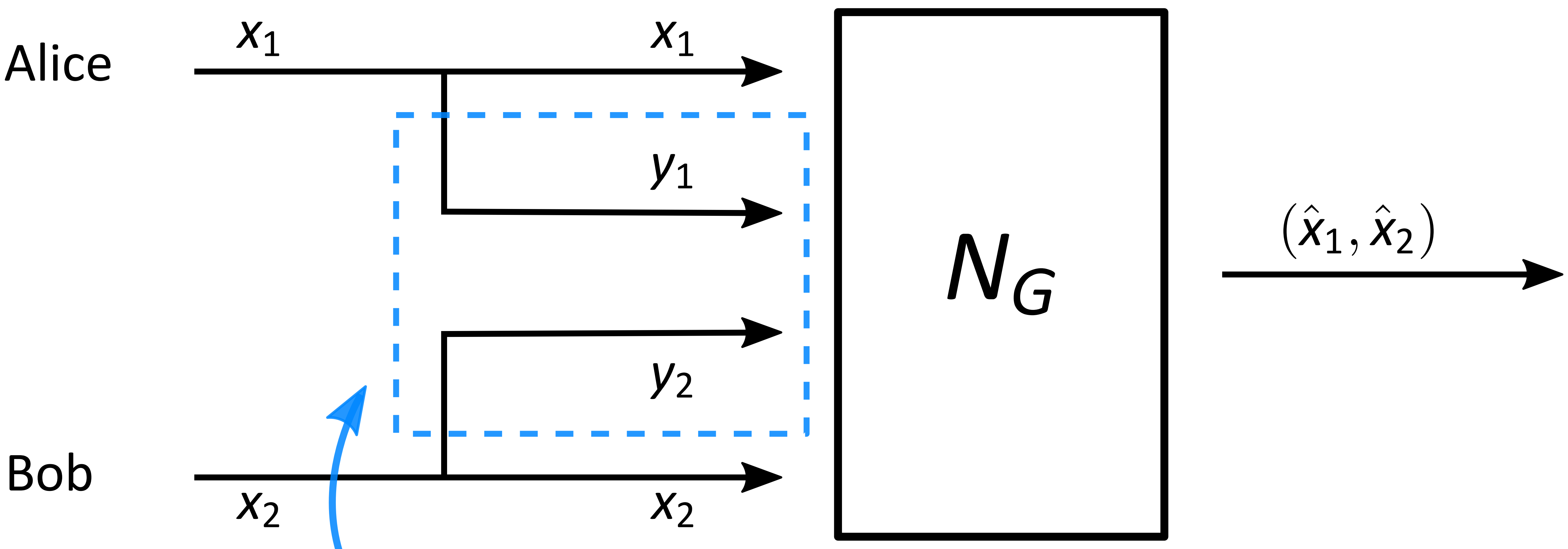
Inputs: question-answer pairs $(x_1, y_1; x_2, y_2)$

Output: question pair $(\hat{x}_1, \hat{x}_2)$



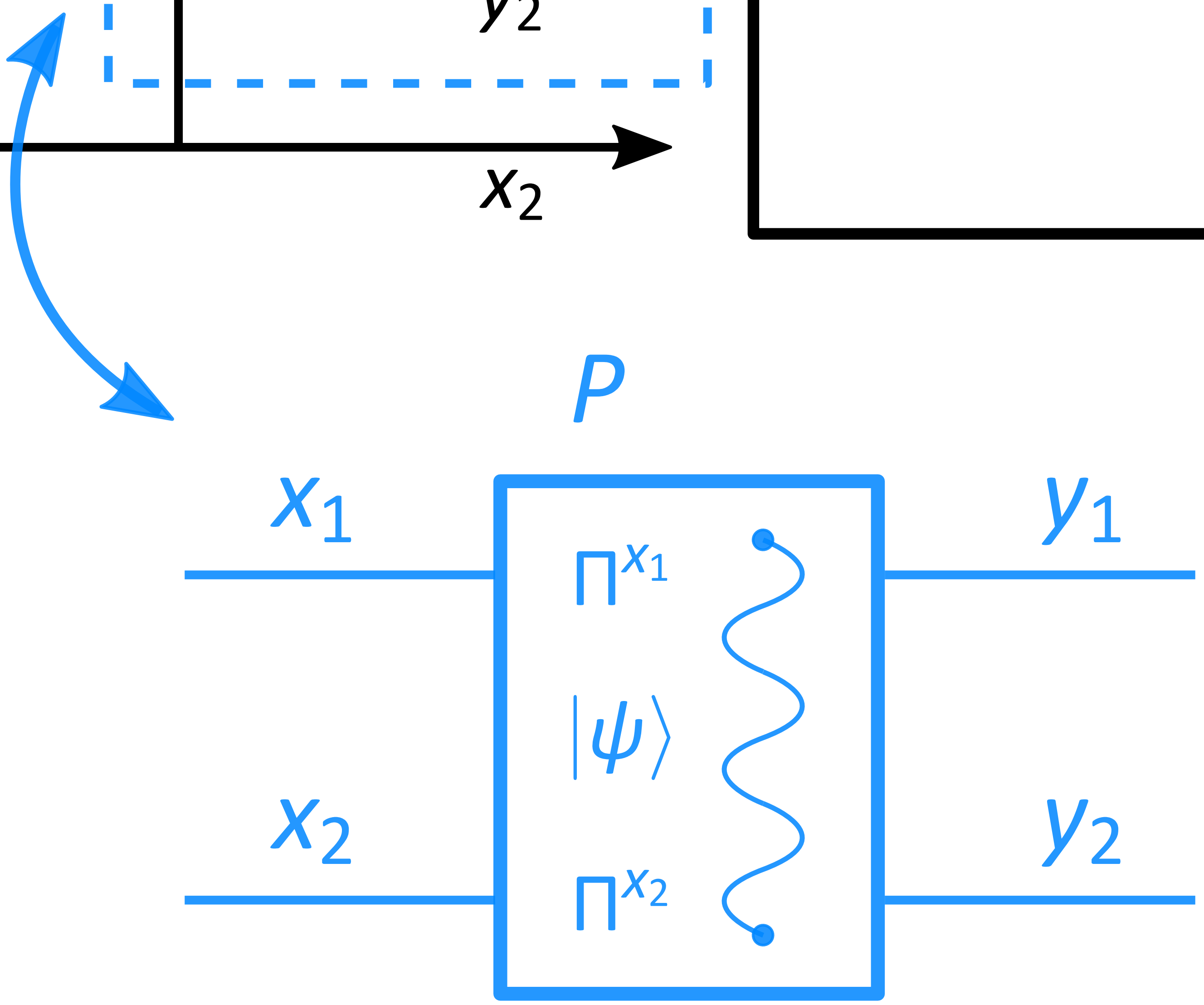
Inspired by [Quek & Shor, PRA 95.5 (2017)].

Entanglement assistance for MACs



If $(x_1, y_1, x_2, y_2) \in W$,
then $\hat{x}_i = x_i$.

If $(x_1, y_1, x_2, y_2) \notin W$,
then $(\hat{x}_1, \hat{x}_2)$ unif. random.



Entanglement-assisted game strategy:

$$P(y_1, y_2 | x_1, x_2) = \langle \psi | \Pi^{x_1}_{y_1} \otimes \Pi^{x_2}_{y_2} | \psi \rangle.$$

for POVMs Π^{x_1} and Π^{x_2}.

Sum rate of a non-local game MAC

Let $G = (\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2, W)$ be a non-local game and N_G the MAC derived from it.

Lemma

Let $p_L = \Pr\{(x_1, y_1, x_2, y_2) \notin W)\}$ be the **losing probability**, and set $Z = (\hat{X}_1, \hat{X}_2)$.

Then $R_1 + R_2 \leq I(X_1Y_1X_2Y_2; Z) = H(Z) - p_L(\log|\mathcal{X}_1| + \log|\mathcal{X}_2|)$.

RHS is maximal when:

1) $H(Z) = \log|\mathcal{X}_1| + \log|\mathcal{X}_2|$;
   only possible with sampling x_i uniformly at random!
2) $p_L = 0$.

Problem

For a non-local game G with classical value $\omega(G) < 1$ players **cannot** win on all questions!

Sum rate of a non-local game MAC

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Main result: No-Go theorem for classical strategies

Let G be a non-local game with classical value $\omega(G) < 1$. Then for the MAC N_G,

$$R_1 + R_2 < \log|\mathcal{X}_1| + \log|\mathcal{X}_2|.$$

Sum rate of a non-local game MAC

Let $G = (\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2, W)$ be a non-local game and N_G the MAC derived from it.

Lemma

Let $p_L = \Pr\{(x_1, y_1, x_2, y_2) \notin W)\}$ be the **losing probability**, and set $Z = (\hat{X}_1, \hat{X}_2)$.

Then $R_1 + R_2 \leq I(X_1 Y_1 X_2 Y_2; Z) = H(Z) - p_L(\log|\mathcal{X}_1| + \log|\mathcal{X}_2|)$.

Main result: perfect sum rate with entanglement

If $\omega^*(G) = 1$, then the **perfect** quantum strategy can be used to **achieve** $(R_1, R_2) = (\log|\mathcal{X}_1|, \log|\mathcal{X}_2|)$ by drawing (x_1, x_2) uniformly at random.

$$\Rightarrow R_1 + R_2 = \log|\mathcal{X}_1| + \log|\mathcal{X}_2|$$

Entanglement helps in a classical task

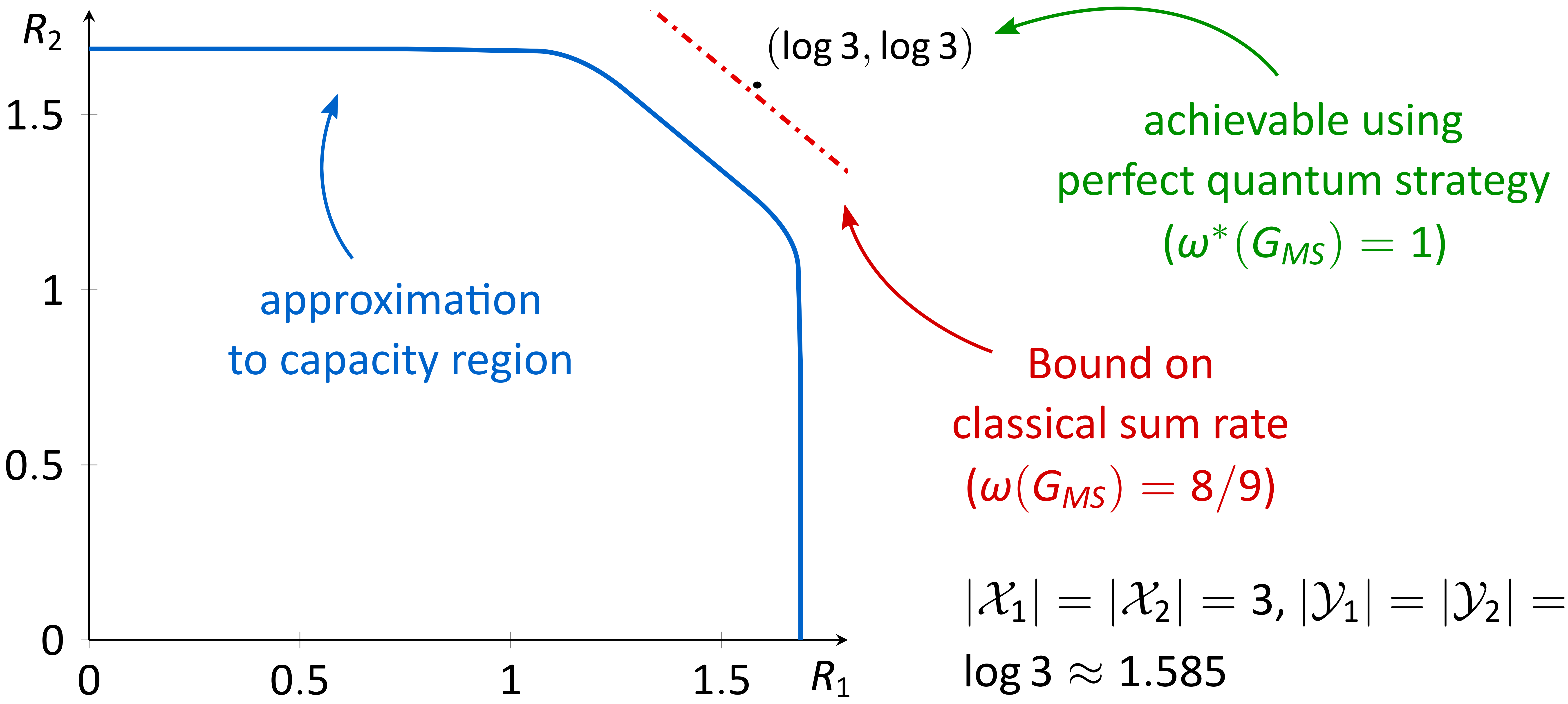
Summary of main result

There are multiple access channels for which the unassisted capacity region and the entanglement-assisted capacity region are **strictly separated**.

In other words: Entanglement shared between senders helps in a **strictly classical** coding task!

Remarkable, because entanglement does not boost (asymptotic) capacity of single-sender-single-receiver channels.

Example: Magic square game channel



$|\mathcal{X}_1| = |\mathcal{X}_2| = 3$, $|\mathcal{Y}_1| = |\mathcal{Y}_2| = 8$

$\log 3 \approx 1.585$

Further results

Main result: If $\omega(G) < 1$ for a non-local game G and a **certain set of strategies**, then $R_1 + R_2 < \log|\mathcal{X}_1| + \log|\mathcal{X}_2|$ for the corresponding MAC N_G.

Unbounded entanglement

There exists a MAC N_G for which the rate point $(\log|\mathcal{X}_1|, \log|\mathcal{X}_2|)$ is **only achievable** using **infinite-dimensional** entangled strategies. [Slofstra and Vidick, Ann. H. Poincare 19.10 (2018)]

There is a family of channels $\{N_G\}_G$ for which it is **undecidable** whether $(\log|\mathcal{X}_1|, \log|\mathcal{X}_2|)$ can be achieved. [Slofstra, Forum Math. Pi 7 (2019)]

NP-hardness

For a given MAC N it is **NP-hard** to decide whether the rate point $(\log|\mathcal{X}_1|, \log|\mathcal{X}_2|)$ belongs to the capacity region (up to $O(n^{-3})$). [Håstad, J. ACM 48.4 (2001)]

Open questions

Information-theoretic

- Can we improve sum rate bound to get "true" separation?
- Formula for the entanglement-assisted capacity region?
- What about arbitrary (three-way) entanglement assistance?

Optimization-theoretic

- Efficiently computable outer bounds for capacity region of MAC?
- Efficient optimization over (bilinear) quantum strategies?
- Can entanglement boost the capacity of arbitrary MACs?

Thanks for your attention!