Dephrasure channel and superadditivity of coherent information

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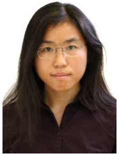

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(IQC, UWaterloo)

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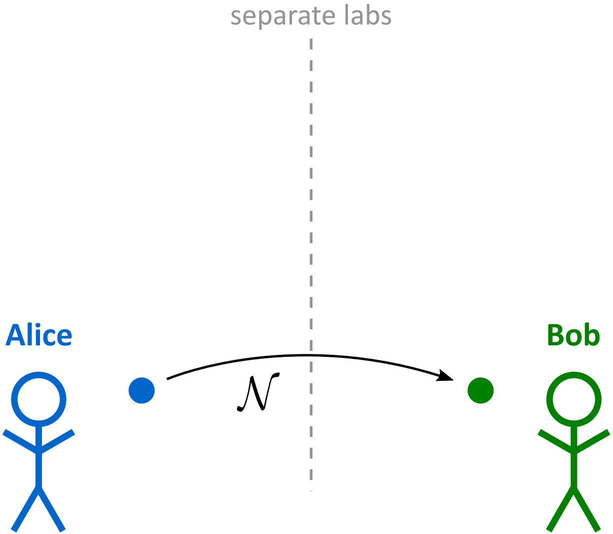
Quantum information transmission



Quantum channel $\mathcal{N}: A \to B$

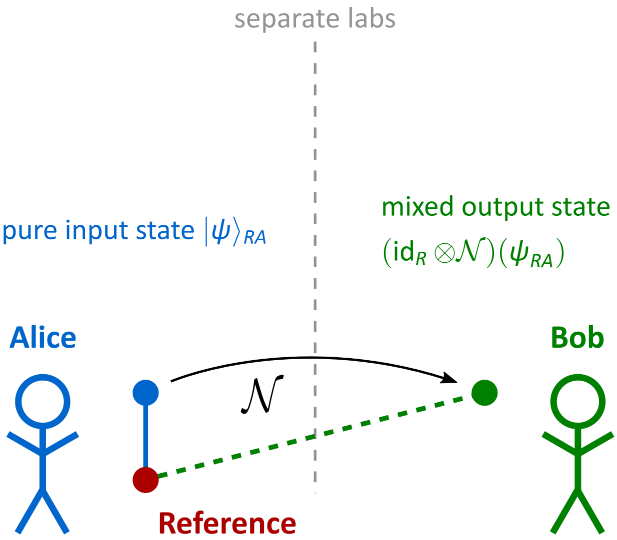
Goal:

Transmit quantum information from Alice to Bob.

Strategy:

Share (mixed) entangled state via $\mathcal{N}$ and distill EPR pairs using local operations and **forward** classical communication $A \to B$.

Quantum information transmission



Rate of the distillation protocol determined by **coherent information**

$$\mathcal{I}_c(\psi_{RA}, \mathcal{N}) := S(\mathcal{N}(\psi_A)) - S(\mathrm{id}_R \otimes \mathcal{N}(\psi_{RA})).$$

Optimizing over **quantum codes** ψ:

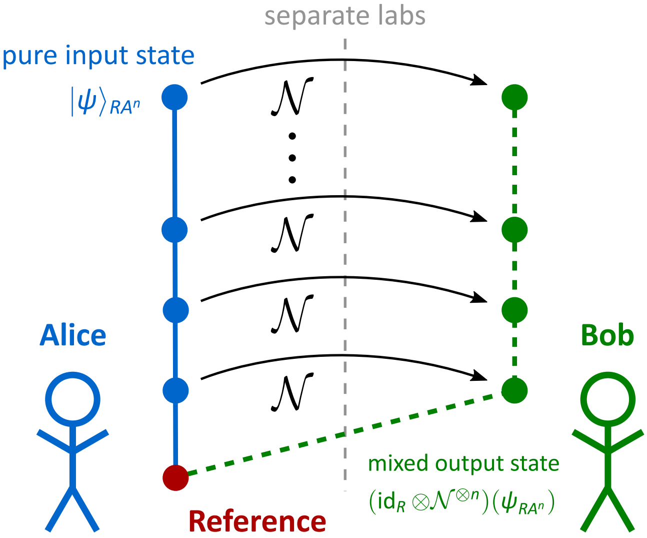
Channel coherent information

$$\mathcal{I}_c(\mathcal{N}) := \sup_\psi \mathcal{I}_c(\psi_{RA}, \mathcal{N}).$$

Can we achieve more?

[Devetak 2005; Devetak, Winter 2005]

Quantum information transmission



Idea: Use n channels in parallel to share multipartite state $|\psi\rangle_{RA^n}$.

Distillation rate: $\frac{1}{n}\mathcal{I}_c(\psi_{RA^n}, \mathcal{N}^{\otimes n})$

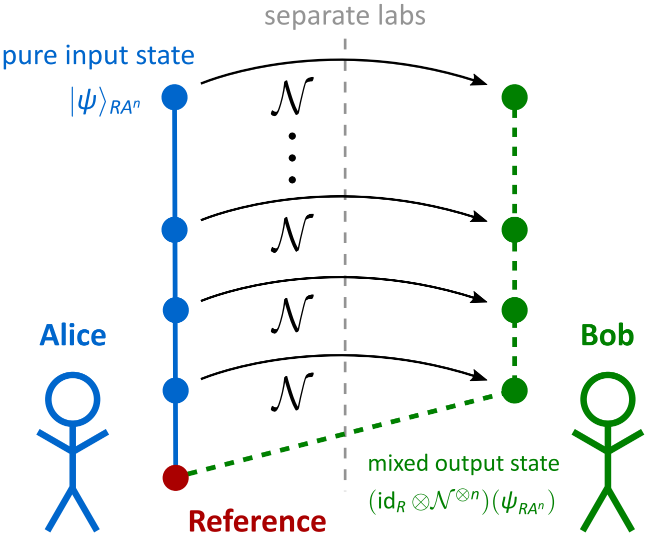
For certain $\mathcal{N}$ and ψ,

$$\frac{1}{n}\mathcal{I}_c(\psi_{RA^n}, \mathcal{N}^{\otimes n}) > \mathcal{I}_c(\mathcal{N}).$$

This is called **superadditivity** of coherent information.

[DiVincenzo et al. 1998]

Quantum information transmission



Quantum capacity:

$$Q(\mathcal{N}) = \sup_{n\in\mathbb{N}} \frac{1}{n} \mathcal{I}_c(\mathcal{N}^{\otimes n})$$

Good: Superadditivity can boost achievable rates, $Q(\mathcal{N}) > \mathcal{I}_c(\mathcal{N})$.

Bad: In general, quantum capacity **intractable to compute**.

Challenge: Find good codes achieving superadditivity.

[Lloyd 1997; Shor 2002; Devetak 2005]

Depolarizing channel

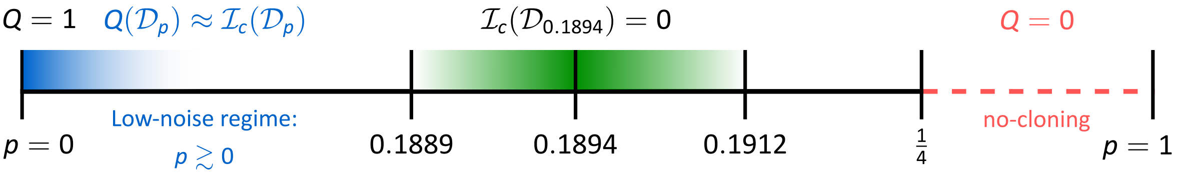
[DiVincenzo et al. 1998]
[Smith, Smolin 2007]
[Fern, Whaley 2008]
[Sutter et al. 2017]
[FL, Leung, Smith 2018]

$$\mathcal{D}_p(\rho) := (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z) \quad \text{for } p \in [0,1].$$

Superadditivity known for $0.1889 \leq p \leq 0.1912$.

Achieved by **repetition codes** $|0\rangle_R \otimes |0\rangle_A^{\otimes n} + |1\rangle_R \otimes |1\rangle_A^{\otimes n}$
and (Shor-like) concatenated codes.

Largest magnitude of superadditivity $\sim 10^{-3}$.



Outline

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Introducing: the dephrasure channel

Dephrasure channel **(dephasing + erasure)**

For $p, q \in [0, 1]$,

$$\mathcal{N}_{p,q}(\rho) := (1-q)\,[(1-p)\rho + pZ\rho Z] \oplus q\,\mathrm{Tr}(\rho)|e\rangle\langle e|.$$

- First dephase the input in Z-basis with probability p, then erase with probability q.
- Simple form of complementary channel to environment:

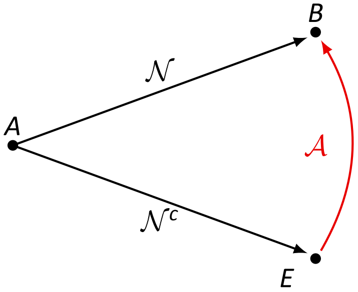
$$\mathcal{N}^c_{p,q}(\rho) = q\,\rho \oplus (1-q) \sum_{x=0,1} \langle x|\rho|x\rangle |\varphi_p^x\rangle\langle\varphi_p^x|$$

  where $|\varphi_p^x\rangle = \sqrt{1-p}\,|0\rangle + (-1)^x\sqrt{p}\,|1\rangle$. $\langle\varphi_p^0|\varphi_p^1\rangle \neq 0$

- Both dephasing and erasure channel are **well understood**, but taken together **interesting things happen**.

Antidegradability of dephrasure channel

A channel $\mathcal{N}$ is called **antidegradable**, if the environment gets enough information from the complementary channel to recover Bob's output.



antidegradable:

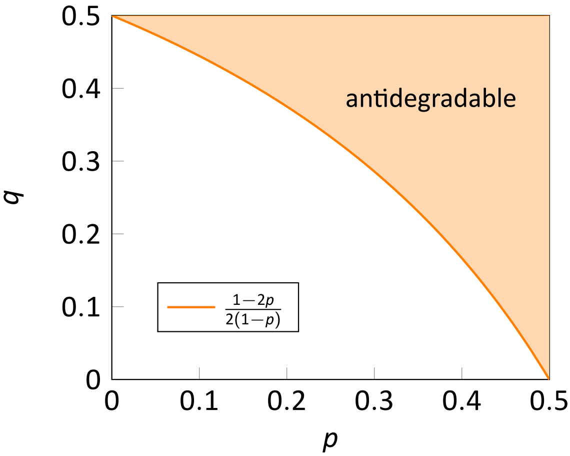
$\exists \mathcal{A}: E \to B$ s.t.

$\mathcal{N} = \mathcal{A} \circ \mathcal{N}^c$

Antidegradable channels cannot transmit quantum information due to **no-cloning theorem**.

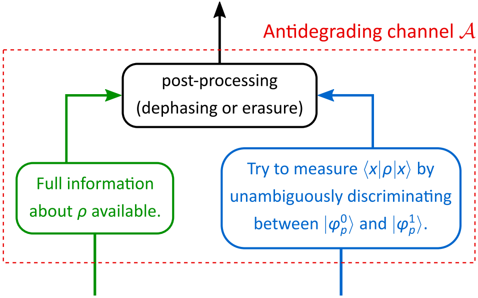
Goal: Determine region of antidegradability of dephrasure channel $\mathcal{N}_{p,q}$ in (p, q)-plane.

Antidegradability of dephrasure channel



Need: channel $\mathcal{A}$ such that $\mathcal{N}_{p,q} = \mathcal{A} \circ \mathcal{N}^c_{p,q}$

$$\mathcal{N}_{p,\,q}(\rho) := (1-q)((1-p)\rho + pZ\rho Z) \oplus q\,\mathrm{Tr}(\rho)|e\rangle\langle e|$$



$$\mathcal{N}^c_{p,\,q}(\rho) = q\,\rho \oplus (1-q) \sum_{x=0,\,1} \langle x|\rho|x\rangle |\varphi^x_p\rangle\langle\varphi^x_p|$$

This scheme works if $\frac{q}{1-q} \geq 1-2p$.

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Single-letter coherent information

- **Goal:** Determine $\mathcal{I}_c(\mathcal{N}_{p,q}) = \sup_\psi (S(\mathcal{N}_{p,q}(\psi_A)) - S(\mathcal{N}^c_{p,q}(\psi_A)))$.

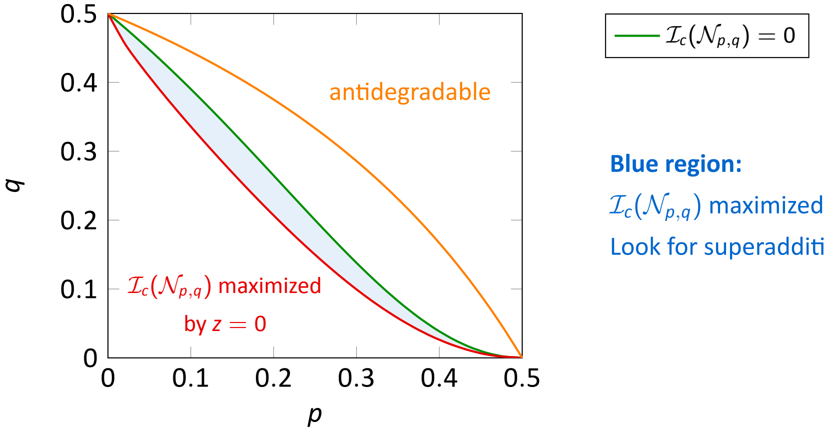
- Calculus: optimal state diagonal in Z-basis, $|\varphi\rangle \sim \sqrt{1+z}\,|0\rangle_R|0\rangle_A + \sqrt{1-z}\,|1\rangle_R|1\rangle_A$.

- $\mathcal{I}_c(\mathcal{N}_{p,q}) = \max_z \left\{ (1-2q)\, S\begin{pmatrix} (1+z)/2 & 0 \\ 0 & (1-z)/2 \end{pmatrix} - (1-q)\, S\begin{pmatrix} 1-p & z\sqrt{p(1-p)} \\ z\sqrt{p(1-p)} & p \end{pmatrix} \right\}$

- Positive for all $q \leq \dfrac{(1-2p)^2}{1+(1-2p)^2}$.

Single-letter coherent information

$$\mathcal{I}_c(\mathcal{N}_{p,q}) = \max_z \left\{ (1-2q)\, S\begin{pmatrix} (1+z)/2 & 0 \\ 0 & (1-z)/2 \end{pmatrix} - (1-q)\, S\begin{pmatrix} 1-p & z\sqrt{p(1-p)} \\ z\sqrt{p(1-p)} & p \end{pmatrix} \right\}$$



Blue region:

$\mathcal{I}_c(\mathcal{N}_{p,q})$ maximized by $z \neq 0$.

Look for superadditivity here!

Superadditivity of coherent information

- First thing to try... **weighted repetition code:** For $\lambda \in [0,1]$,

$$|\varphi_n\rangle_{RA^n} = \sqrt{\lambda}\,|0\rangle_R|0\rangle_A^{\otimes n} + \sqrt{1-\lambda}\,|1\rangle_R|1\rangle_A^{\otimes n}$$

- For $n = 1$, this is the optimal single-letter code ($\lambda = (1+z)/2$).

- $\mathcal{N}_{p,q}^{\otimes n} = ((1-q)\mathcal{Z}_p + q\,\mathrm{Tr}(\cdot)|e\rangle\langle e|)^{\otimes n}$:

  sum of channels of the form $\mathcal{Z}_p^{\otimes k} \otimes (\mathrm{Tr}(\cdot)|e\rangle\langle e|)^{\otimes n-k}$.

- Coherent information splits up into different erasure patterns, since

$$S(\textstyle\bigoplus_i p_i\rho_i) = \sum_i p_i S(\rho_i) + H(\{p_i\}).$$

- Repetition code: all **partial erasures cancel**, compute action of $\mathcal{Z}_p^{\otimes n}$ on φ_n.

Superadditivity of coherent information

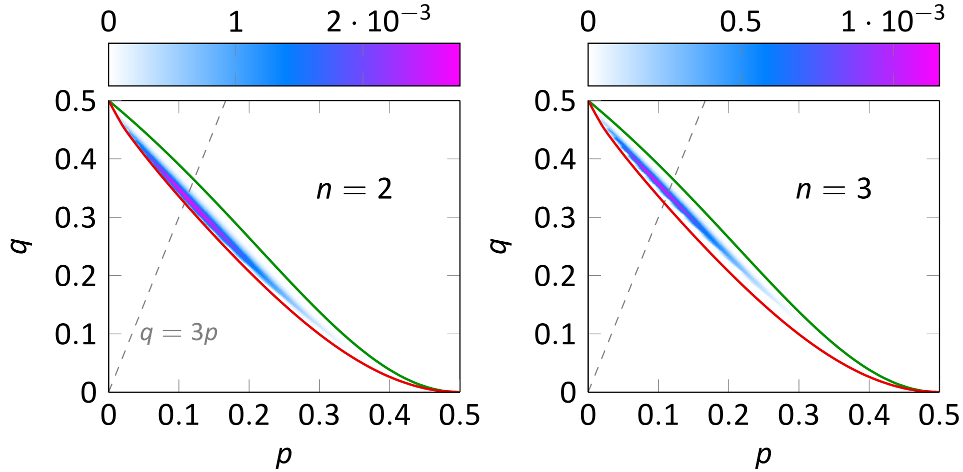
- Formula for repetition code ($\varphi_n = \varphi_n(\lambda)$):

$$\mathcal{I}_c(\varphi_n, \mathcal{N}_{p,q}^{\otimes n}) = \max_{\lambda} \left\{ ((1-q)^n - q^n)\, h(\lambda) - (1-q)^n \left(1 - u \operatorname{artanh} u - \tfrac{1}{2}\log\left(1-u^2\right)\right) \right\}.$$

- $u = u(\lambda, p, n) = \sqrt{1 - 4\lambda(1-\lambda)(1-(1-2p)^{2n})}$.
- Binary entropy: $h(\lambda) = -\lambda \log \lambda - (1-\lambda)\log(1-\lambda)$.
- Threshold of $\mathcal{I}_c(\varphi_n, \mathcal{N}_{p,q}^{\otimes n})$ is the same for all $n \in \mathbb{N}$: $q = \dfrac{(1-2p)^2}{1+(1-2p)^2}$

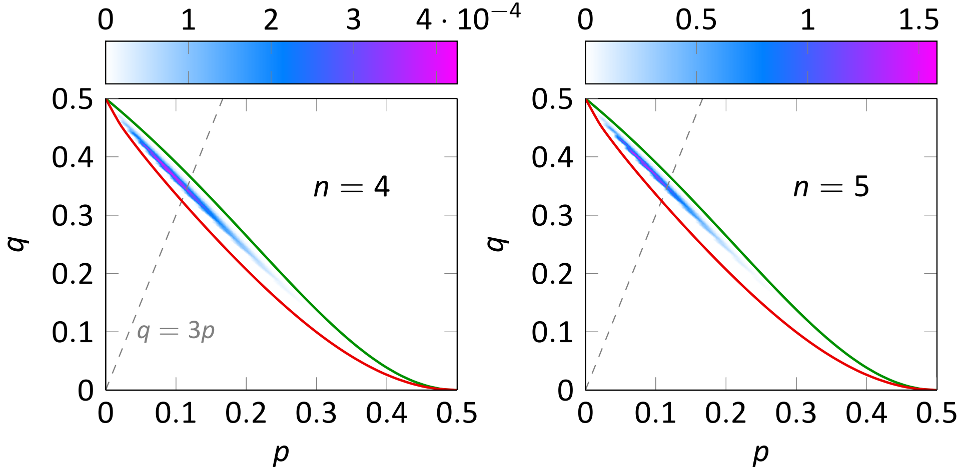
Superadditivity of coherent information

Plot for $n = 2, 3$ of the non-negative part of $\frac{1}{n} \max_{\lambda} \mathcal{I}_c(\varphi_n, \mathcal{N}_{p,q}^{\otimes n}) - \mathcal{I}_c(\mathcal{N}_{p,q})$



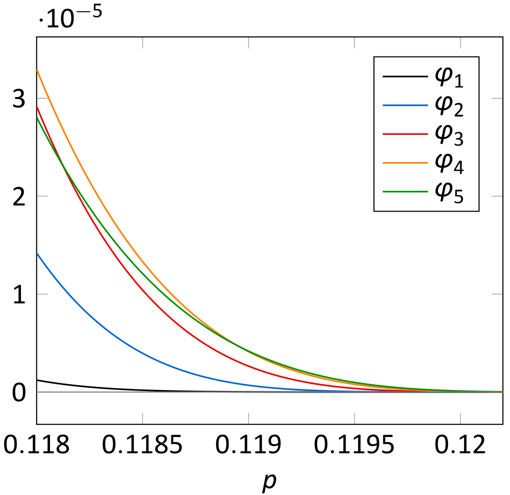
Superadditivity of coherent information

Plot for $n = 4, 5$ of the non-negative part of $\frac{1}{n} \max_{\lambda} \mathcal{I}_c(\varphi_n, \mathcal{N}_{p,q}^{\otimes n}) - \mathcal{I}_c(\mathcal{N}_{p,q})$



Superadditivity of coherent information

Plot for $\frac{1}{n}\mathcal{I}_c(\varphi_n, \mathcal{N}_{p,3p})$ along diagonal $(p, 3p)$



Superadditivity of coherent information

- We also found more elaborate, **non-diagonal** codes achieving superadditivity, e.g.:

$$|\chi_3\rangle := |00\rangle_R \otimes |00\rangle \otimes |\psi_1\rangle + |11\rangle_R \otimes |11\rangle \otimes |\psi_1\rangle \\ + |01\rangle_R \otimes |01\rangle \otimes |\psi_2\rangle + |10\rangle_R \otimes |10\rangle \otimes X|\psi_2\rangle,$$

  for some pure states $|\psi_i\rangle$.

- We also found good codes using a **neural network state ansatz**, outperforming all other codes. **SQuInT Poster #33**, [Bausch, FL; arXiv:1806.08781]

- Haven't found codes increasing the single-letter threshold yet.

- Not clear whether optimal codes for $n \geq 2$ are diagonal in Z-basis (true for $n = 1$).

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Private information transmission

- **Private capacity** $P(\mathcal{N})$: highest rate of faithful **private classical communication** between Alice and Bob, [Devetak 2005]

$$P(\mathcal{N}) = \sup_{n\in\mathbb{N}} \frac{1}{n}\mathcal{I}_p(\mathcal{N}^{\otimes n}),$$

with the **private information** $\mathcal{I}_p(\mathcal{N}) := \max_{\{p_x,\rho_x\}} [I(X;B)_{\mathcal{N}(\rho)} - I(X;E)_{\mathcal{N}^c(\rho)}]$.

- Private information can also be **superadditive**, $\frac{1}{n}\mathcal{I}_p(\mathcal{N}^{\otimes n}) > \mathcal{I}_p(\mathcal{N})$. [Smith et al. 2008]

- Quantum information transmission is necessarily private:

$$\begin{array}{ccc} Q(\mathcal{N}) & \leq & P(\mathcal{N}) \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\ \mathcal{I}_c(\mathcal{N}) & \leq & \mathcal{I}_p(\mathcal{N}) \end{array}$$

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superadditivity of $\mathcal{I}_c(\cdot)$ and $\mathcal{I}_p(\cdot)$

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superadditivity of $\mathcal{I}_c(\cdot)$ and $\mathcal{I}_p(\cdot)$

separation of capacities

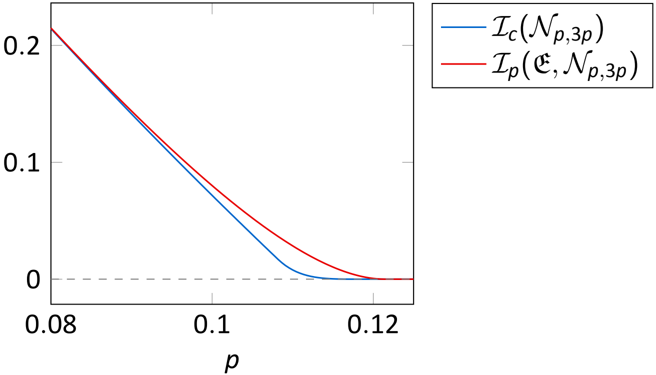
[Horodecki et al. 2005]
[Leung et al. 2014]

Separation of private and coherent information

- Numerical investigations suggest the following private ensemble $\mathfrak{E}$ is optimal:

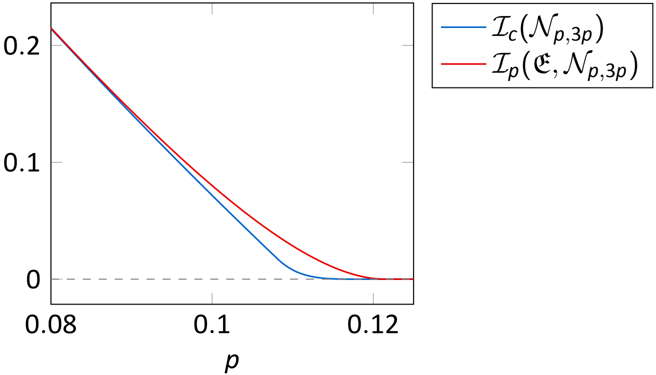
$$p_1 = \tfrac{1}{2}, \quad \rho_1 = \lambda|+\rangle\langle+| + (1-\lambda)|-\rangle\langle-| \qquad |\pm\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$$

$$p_2 = \tfrac{1}{2}, \quad \rho_2 = \lambda|-\rangle\langle-| + (1-\lambda)|+\rangle\langle+|$$



Separation of private and coherent information

- **Superadditivity of private information?** $\mathcal{I}_p(\mathcal{N}_{p,q}^{\otimes n}) > n\mathcal{I}_p(\mathcal{N}_{p,q})$?
- **Separation of capacities?** $P(\mathcal{N}_{p,q}) > Q(\mathcal{N}_{p,q})$?



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Conclusion

- Superadditivity is a poorly understood phenomenon.
- **Boosts** communication rates.
- Renders quantum channel capacities **intractable to compute**.
- **Dephrasure channel:** $\mathcal{N}_{p,q}(\rho) = (1-q)\left[(1-p)\rho + pZ\rho Z\right] + q\,\mathrm{Tr}(\rho)|e\rangle\langle e|$.
- Particularly simple channel exhibiting **substantial superadditivity**.
- Excellent toy model to study superadditivity and quantum channel capacities.

Open questions

- New codes to increase threshold of single-letter and repetition codes?
- Are Z-diagonal codes optimal (multi-letter)?
- What is the optimal private ensemble (single-letter)?
- Superadditivity of private information?
- Separation of quantum and private capacities?
- What about other capacities of the dephrasure channel (e.g. assisted capacities)?

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Thank you for your attention!